Stuck Behind the Math: Just How Helpful Can One Expect Technology to be in the Economics Classroom?

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Abstract

This paper studies the efficacy and motivation for using technology when explaining mathematical concepts in economics pedagogy. It links practice with theory by examining the use of technology in light of current literature on active and cooperative learning in economics in conjunction with extensive research in mathematics education concerning the effectiveness of technology in teaching. Suggestions on how to implement teaching strategies with computer technology are discussed within the context of the computer algebra system, *Maple*. A variety of specific applications using *Maple* technology are provided in an online appendix.

Introduction

Few other disciplines are poised to have their pedagogies so radically affected by technology. In just 15 years the use of technology as a pedagogical tool in economics has become ubiquitous due to the increasing accessibility of information via the web and the plethora of software available for analysing this information (Walbert and Ostrosky, 1997; Chizmar and Walbert, 1999). Indeed, the personal computer is primed to become the new economics chalkboard of the 21st century. Most obviously, the use of spreadsheets in teaching econometrics and symbolic algebra software in teaching economic theory has allowed instructors to become more efficient while enhancing the learning experience (Murray, 1999; Walbert and Ostrosky, 1997).

Although cognitive research in psychology and education suggests that active and cooperative learning strategies such as those implementing technology enhance both student learning and interest, survey results from Becker and Watts (1995, 1996, 2001) indicate that the transition to technology-based instruction has not been widely adopted by economics college instructors. According to Becker and Watts, instruction in the typical undergraduate economics classroom is still based primarily on lectures, with the chalkboard and text assignments as complements. This is particularly true of courses in which conceptual ideas and more rigorous mathematical models are stressed, most notably, the intermediate micro and macroeconomic theory courses. We believe that the lack of uniformity in transitioning to technology-based pedagogy stems from an incomplete understanding of the benefits of technology in addition to a perception that the start-up costs associated with its incorporation are high. Our paper addresses both of these issues.

First, we use pedagogical research in economics and mathematics to understand how technology aids learning and to discuss why some classroom objectives may be achieved more easily when technology is used to complement lecture material. Although instructors of economics have begun to incorporate insights from learning theory into teaching pedagogy (notably Goffe and Sosin, 2005), existing research in psychology and education has much more to offer when attempting to assess the benefits of technology. Given the degree to which economic theory has come to rely on mathematical tools, it follows that the abundance of research in mathematics education is particularly appropriate. Previous papers on the use of technology in economics pedagogy have primarily (though not exclusively) focused on how it can be used to generate solutions to specific problems in economics (Boyd, 1998 or Cahill and Kosicki, 2000a and 2000b). Our paper expands the literature on economic pedagogy by linking the conclusions of current research in mathematics education on the general application of technology to the literature on active and cooperative learning in economics.

Second, we use the theoretical implications from mathematics and economics education to illustrate how technology may enhance learning mathematical concepts in the intermediate economics classes. Because the intermediate courses rely heavily on mathematical theory, these courses are an ideal place in which to demonstrate the usefulness of technology in teaching mathematical concepts. *Maple*, a symbolic software program, is incorporated in the classroom to reinforce mathematical principles. We offer teaching tips and suggestions to instructors implementing this approach in order to increase the probability of a successful experience and minimise start up costs.

The sections that follow discuss why, when and to what extent using technology may be beneficial in the economics classroom. First we discuss how and why *Maple*

software is appropriate for intermediate economic theory courses. We then examine research in economic pedagogy in more detail. This literature suggests the importance of using active or cooperative learning techniques in order to accomplish learning objectives in the intermediate theory courses. This is followed by a discussion of the role of technology, particularly *Maple*, in teaching mathematical concepts. Economics theory depends on the language of mathematics. This creates an important and obvious link between economics pedagogy and research in mathematics education.

The concluding section offers suggestions on how to integrate technology in order to maximise its effectiveness. An appendix containing the *Maple* code and colour output from eight specific examples in both intermediate microeconomics and macroeconomics is available on the author's website, http://faculty.bellarmine.edu/fraymond/teaching.asp.[1]

Why use *Maple* as a pedagogical tool in economics?

Comparison to other software

Maple is a symbolic computation, or computer algebra system (CAS) that can solve systems of algebraic equations, perform numerical computations, and provide 2D and 3D colour graphics. *Maple* graphics and code can be embedded directly into *Microsoft Word* or *Excel*.

It is natural to wonder why one would choose *Maple* over other computer algebra systems and to ask how *Maple* software compares to other software. The development of computer algebra systems began in the late 1960s with the software becoming available for public use in the early 1970s (Fateman, 1972). Some of the earliest CAS software, such as *BERGMAN[1]* and *REDUCE*, is still in use today. Many computer algebra systems were developed with specific, though limited, capabilities. However, there are a number of computer algebra systems that perform a wide range of mathematical computations and provide multidimensional graphics.

Among the most versatile CAS software are *DERIVE*, *MuPAD*, *Mathematica* and *Maple*.[2] There are also choices such as *MATLAB*, designed primarily for numerical computations, which can be interfaced with *Maple* to provide a complete CAS package. These computer algebra systems tend to be used in academic settings and strive to be user-friendly. Because of the ability to perform symbolic manipulations, to find complicated algebraic solutions with high degrees of accuracy and to provide high-quality graphics, these packages are used when teaching algebra, trigonometry and calculus.

DERIVE was first released in 1988 by Texas Instruments, but is no longer in production. *DERIVE* is a symbolic algebra program with 2D and 3D graphing, but is compatible only with the Windows operating system. It is a program capable of supporting prealgebra, algebra, trigonometry, precalculus and calculus courses. Texas Instruments is currently working on a next generation computer algebra system. *Mathematica*, first released in 1988, brought computer algebra into mainstream use. For 20 years, the creator of this software, Wolfram Research, has been a consistent leader in computer algebra research. *Maple* was first developed by the Symbolic Computation Group at the University of Waterloo in Ontario, Canada in 1980. *Maple* was originally released in 1985 by Waterloo *Maple*, Inc. *Maple* and *Mathematica* have become extremely popular in both industrial applications and academia. *MuPAD* was originally developed in 1992 by the *MuPAD* research group at the University of Paderborn. However, since 1997 it has been jointly developed by SciFace Software in cooperation with the *MuPAD* research group. The *MuPAD* kernel is also used in other more general scientific and mathematical software. It recently replaced the *Maple* kernel within the software, *MathCAD*.

Mathematica, *Maple* and *MuPAD* are mathematics engines that perform the symbolic computations fundamental to prealgebra, algebra, trigonometry and calculus. Each can evaluate, factor, combine, expand and simplify expressions that contain integers, fractions, and real and complex numbers. They can also be used to evaluate integrals and derivatives, perform matrix and vector operations, find standard deviations, and perform many other more complex computations. In addition, one can create 2D and 3D plots of polynomials, trigonometric functions and exponentials, and you can create animated 2D and 3D graphics. *Mathematica* and *Maple* incorporate a full high-level programming language that interfaces to other languages and with Excel. (*Maple* interfaces with C, Fortran, Java, Matlab, and Visual Basic.) All three of these CAS packages run on most all main operating systems including Windows and MAC-OS.

Shone (1997) states that, for the mathematical economist, *Mathematica* is highly sophisticated and able to solve a broad range of problems. However, he acknowledges that *Maple* is easier to use. According to Pattee (1995), because *Maple* and *Mathematica* are used extensively in research and industry, there is a wealth of literature, sample worksheets, advice and hints available to users (Abell and Braselton, 2003, 2005). Although both programs offer similar benefits, *Maple* is generally recognised as more efficient in solving differential equations, while Mathematica's interface is often considered preferable for demonstrations. However, it is important to note that competition in the past decade has driven each product to strive to perfect any perceived shortcomings. Thus, for the vast majority of users, today the two products are virtually interchangeable.

Tonisson (1999) compared the software packages *DERIVE*, *Maple*, *Mathematica*, and *MuPAD*. The purpose of this paper was to compare aspects related to one-step and step-by-step solutions using the four different CAS packages. In one-step problems, *Maple* and *Mathematica* found correct solutions in each example, while *DERIVE* and *MuPAD* were successful in most, but not all, examples. In step-by-step problems, each software package approached the selected problems differently and provided varying degrees of information for students. *Maple* and *Mathematica* were found to address step-by-step solutions more explicitly than *DERIVE* and *MuPAD*. *Maple* includes some information about step-by-step solutions in the student package introduction and *Mathematica* has a demo-file on step-by-step differentiation. In general, CAS packages hesitate to 'show' steps because they use other algorithms than those used in student textbooks and the intermediate results may be confusing for students (Tonisson, 1999).

Seven CAS software packages, *AXIOM*, *DERIVE*, *MACSYMA*, *Maple*, *Mathematica*, *MuPAD*, and *REDUCE*, were compared in 'A Critique of Mathematical Abilities of CA Systems' (Wester, 1999). This was a comprehensive comparison of seven all-purpose CAS packages and their performance on 542 mathematical problems. The problems were divided into 30 different sections and presented in order of increasing logical difficulty. For each problem, the article summarised a variety of issues including whether each of the CAS software packages (1) attempted to get a solution, (2) was able to get a solution after one hour of CPU time, (3) produced a small error in a determined solution, (4) had some success, but also some mistakes, (5) had an answer that was mostly wrong, but still had a few positive elements, or (6) had complete success (Wester, 1999). Although the seven different CAS packages had varying degrees of success across the different levels of mathematical problem solving, *Maple*, *Mathematica*, and *MuPAD* showed consistent levels of success with *Maple* and *Mathematica* showing the highest level of consistency in ability to solve problems successfully.

In sum, the development of CAS software has been on-going for more than three decades. Many are compatible with both Windows and Mac operating systems (and other systems) and most provide the symbolic manipulation to facilitate problem solving needed in prealgebra, algebra, trigonometry and calculus courses, and provide the graphing capabilities to illustrate 2D and 3D images. *Maple* and *Mathematica* are each other's primary competitors and are the current market leaders in CAS software. In comparisons, these two CAS software packages have been shown to be the most consistent and successful in solving a wide range of problems. *Maple* and *Mathematica* appear to perform similarly and it would be reasonable to choose either for academic applications. Early on, Waterloo *Maple* appears to have focused a bit more on the academic market, gaining popularity by

offering lower academic prices. In addition, *Maple* is also equipped with an ever-evolving statistical package. Thus, the choice of *Maple* as the pedagogical tool for solving mathematical economics problems is a reasonable one.

In addition, there are significant advantages to using symbolic computation software instead of spreadsheet programs such as *Excel*, *Lotus* and *QuatroPro*. Spreadsheet programs are specifically designed to manipulate numerical data so they cannot generate purely algebraic results. Because spreadsheet programs are restricted to numerical methods and solutions, symbolic manipulators such as *Maple* and *Mathematica* can be more versatile, even when finding numerical solutions. Additionally, Saunders (1998) states that new learning is more efficient if linked to material students have previously learned. Economics students are often exposed to *Maple* in calculus, further enhancing the efficacy of the software.

Despite its advantages, symbolic computation software also has some limitations. It is limited in its ability to solve relatively complicated non-linear systems, algebraic solutions are not always presented in simplified form, and a spreadsheet may be more efficient if one's primary goal is to generate large sequences of numerical solutions based on a repetitive algorithm. These limitations can sometimes be significant when relying on symbolic computation systems for research purposes. Nevertheless, symbolic computation systems are sophisticated enough to handle a wide variety of applications, including those most often found in the undergraduate curricula in mathematics or mathematical sciences, so that *Maple* can be a powerful undergraduate pedagogical tool when appropriately used.

Technology for a purpose

Recently, many colleges and universities have encouraged instructors to integrate technology more extensively in the classroom. In some courses such as statistics or econometrics the benefits of technology are clear. However, in other courses such as intermediate economic theory, where the benefits are less obvious, the push for technology can also create a 'perverse incentive', as it naturally leads one to ask if the technology is to be incorporated for its own sake or for some other reason. Contrary to its intended outcome, using technology without a specific pedagogical purpose can actually reduce efficiency by making the classroom experience less focused and more confusing. Because teaching technology requires the sacrifice of time typically allocated to more traditional (and not necessarily less important) examples or conceptual issues, it is especially imperative that instructors have a specific objective in mind, as well as an efficient method of integration.

Intermediate economic theory has traditionally been taught conceptually, rather than computationally. Students learned course material by engaging in passive

activities such as note taking and reading. However, the recent pedagogical literature on active learning has prompted instructors to focus more fully on problem-solving approaches. Because it is difficult to construct a wide variety of conceptual problems for the students to complete, there is a natural incentive to develop more computational problems. This has stimulated further discussion about the relative importance of teaching the underlying mathematics, particularly in intermediate microeconomics. In fact, more colleges and universities generally appear to be moving toward requiring one semester of calculus as a prerequisite for intermediate microeconomics.

The debate is probably most relevant for intermediate microeconomics because the majority of the course is based upon the concept of constrained maximisation as applied to consumer choice and production theory. One benefit from using technology for solving algebraic systems is particularly evident in the following example. In consumer choice, the preferences of a representative consumer are defined by a utility (objective) function, $f = f(x, y)$. The consumer maximises utility by choosing appropriate levels of goods x and y with respect to a linear income or budget constraint, $B = P_x x + P_y y$ where P_x and P_y where are respective prices. In most cases, f is assumed to be increasing, concave and at least twice differentiable. Utility is constant along any indifference curve in the indifference map, the set of contour lines associated with f. Since the previous assumptions imply that these indifference curves are concave, utility maximisation occurs where the budget line is tangent to the indifference curve furthest from the origin. The producer problem is defined similarly, with one important conceptual difference. Production is defined by the function $f = f(x, y)$ where x and y are inputs. The production function has many of the same mathematical properties as a utility function. Again, f is assumed to be increasing, concave and twice differentiable. Costs are defined by $P_x x + P_y y = C$ where C is total cost and P_x and P_y are the respective prices of inputs x and y. The conceptual difference distinguishing production is that the contour lines (isoquants) of represent production constraints. The goal is to minimise costs (objective) subject to a convex production constraint. To the perceptive student, this appears to be the reverse of the consumer problem since the contour lines make up the constraint set. However for the average student, distinguishing between the objective and the constraint can cause confusion and lead to misunderstanding, even if the fundamentals of the consumer problem were initially grasped. This subtle, yet conceptually important distinction becomes much more apparent when *Maple* is used to demonstrate the mathematics.

We have demonstrated that there are at least three principal motives for incorporating *Maple* as a pedagogical tool. These include the ability to (i) enhance active and cooperative learning, (ii) illustrate subtleties that are difficult to explain conceptually, and (iii) illustrate solutions to specific problems involving theoretical or conceptual ideas. It is especially beneficial if its use facilitates understanding of material that in the past was considered too complex for a particular audience. Note that this point illustrates an important difference between the benefits of using *Maple* in economics as compared to mathematics pedagogy. As indicated in the mathematics education literature, symbolic manipulation systems allow the mathematics instructor to spend less time on computations and more class time on conceptual issues (enhancing quantity). However, since economics pedagogy has traditionally fostered a conceptual approach, the current trend is to augment pedagogy with more examples that promote active learning (enhancing quality). Thus, both disciplines may use *Maple* as an active learning tool in order to enhance the conceptual component of current pedagogy, although for different reasons.

Integrating technology with pedagogy

One can implement *Maple* as a pedagogical tool in a typical intermediate microeconomics course (not calculus based) as long as the students are familiar with the basic notion of a derivative. Thus, a scientific or business calculus course could be taken concurrently or as a pre-requisite. In our experience, students had typically completed a one semester course in either business or scientific calculus before enrolling in intermediate economic theory, although there was an occasional student taking these courses concurrently. Early in the course a discussion on the importance of 'marginal' concepts in economic theory was used to motivate and illustrate the use of a derivative. The general definition of a derivative was derived in a simple example, followed by examples illustrating the shortcuts used for differentiating powers and logarithms. This required approximately 45 minutes of class time. After completing constrained optimisation in consumer choice, the instructor introduced the concept of the Lagrangian, as an extension of the earlier discussion on derivatives, requiring another 30–40 minutes of class time.

Implementing *Maple* into an intermediate theory classroom also requires two or three sessions in a computer classroom. Although it is most effective if each student has his own computer, pairs of students may share. At or just prior to the start of the session, the instructor must illustrate the connection between the economics studied in class and its mathematical representation on the blackboard. Afterwards, the instructor must translate what he has demonstrated into *Maple* code on the blackboard. The students are initially requested to open a new *Maple* window and simply copy a template as given and run the program. This familiarises the student with *Maple* syntax and allows them to witness the execution of the program while visualising the mathematical concept. Afterwards, the instructor may give either verbal or written instructions to the students, directing them to

alter the template. This process requires students to recognise the specific nature of the amendment and implement the change. For example, when teaching consumer theory, students must recognise the appropriate template changes to make in response to variations in preferences or prices. This reinforces the economic concepts of the utility function and the budget constraint, while also allowing students to see how mathematics provides a specific solution to the problem. Complementing lecture material with *Maple* technology increases the ability to make subtle conceptual differences more evident.

Student reaction to the use of *Maple* technology was initially mixed, but became increasingly positive as familiarity with the software grew. Although some apprehension was expressed when the first *Maple* project was assigned, introducing the software in the computer classroom and providing two to three hard copy example-templates for reference was helpful in building confidence. Once the assignments were complete, students expressed that they better understood the importance of constrained optimisation in economics. Teaching evaluations indicated that the students recognised the *Maple* exercises as no less important than any other portion of the course. The students verbally communicated that the exercises were most helpful as a way to deepen their understanding of challenging theoretical concepts. This is consistent with the criteria that Salemi (2002) uses for selecting topics appropriate for active learning. In particular, he suggests that the topics be relatively important as active learning strategies require more class time than alternative techniques, and also that they involve more difficult concepts.

The importance of active learning techniques in accomplishing learning objectives in intermediate theory

Learning theory and the goals of economic education

As Siegfried (1998) suggests, most economists would agree that the most important goal of economics education is to enable students to develop the ability to 'think like an economist'. This involves the development of problem-solving skills that stress analytical reasoning in order to understand economic relationships, as well as creative skills which enable the student (1) to pose questions, (2) to use appropriate tools and data to answer those questions, and (3) to explain unexpected empirical results. This type of thinking is facilitated by providing numerous opportunities for students to apply economic concepts to a variety of new situations so that they learn how 'to do' economics as well as courses which increase both the breadth and depth of economic knowledge and its relationship to other disciplines.

Saunders' (1998) review of the learning literature developed in psychology and education supports the view that problem-solving skills are enhanced with opportunities for the students to actively apply concepts. He begins with four generally accepted propositions about learning. First, the human mind has a limited capacity to process information. This implies that focusing on a few concepts rather than a broad spectrum of topics is more effective. Second, new material is learned most effectively when it is linked to prior experience. Third, motivation is an important factor in determining how well the material is absorbed, so it is important that concepts appear relevant in order to stimulate and maintain interest. Finally, visual learning experiences are more effective than verbal learning experiences, so learning is more efficient if visual aids are included in pedagogy. Based on these propositions, Saunders defines learning as the acquisition and retention of knowledge and habits of thought in a way that permits them to be employed in a useful way after the initial exposure has been terminated. He argues that thinking like an economist requires that students analyse problems independently, which involves three stages of learning: acquisition, retention and transfer. Acquisition or learning new material is most efficient when the student is clear about what they are supposed to learn and the new material is linked to material the student previously learned.[3] Retention occurs as the student remembers the material and is most efficient when the material is effectively acquired. Transfer or learning to apply the material is accomplished by pedagogical tools that emphasise the characteristics that are common to particular types of problems.[4]

Several pedagogical implications follow from Saunders' research of learning theory. In order to enhance acquisition as well as retention, lectures should be complemented with active[5] (Becker, 1977) and cooperative[6] learning techniques, repetition and quick feedback to the student. Visual aids should be used when possible. In order to promote the student's ability to transfer or apply the material, basic concepts should be repeatedly interpreted and applied to a variety of situations in class and in exercises so that the student perceives the usefulness of the general principles. Exercises should focus on interpretation and application rather than memorisation or computation.

Salemi (2002) provides a particularly strong case for active learning techniques by outlining numerous benefits that stem from their inclusion in the classroom.[7] Benefits include a deeper understanding of course concepts as students work harder in class and collaboratively revise their thinking to make ideas work versus passively interpreting an instructor's lecture. During this process, instructors and students both benefit from feedback. Students receive immediate information from instructors and peers on how well they understand concepts. As instructors monitor

class discussions, they can more effectively guide future discussions to address students' needs. Additional advantages include the ability to address a wider range of student learning styles so that the effectiveness of class time is increased. Salemi argues that active learning works well in classes comprised of slow and fast learners because the faster learners can help explain concepts to the slower learners as class collaboration occurs. This peer interaction is also helpful in teaching students critical skills as 'they learn to listen critically, to question what they do not understand and to challenge when they disagree (p. 728)'. Applying material with active learning techniques increases the likelihood of positive student attitudes about learning by showing the relevance of economics, by motivating students to assume greater responsibility for their learning and by creating incentives to be more prepared to participate. Finally, Salemi argues that active learning classes are more interesting to teach, as classroom discussions are less likely to be limited to preconceived agendas, allowing for more novel approaches.

Intermediate economic theory and active learning

The intermediate theory courses are central in attaining the goals of the economics major. The objectives include the development of an appreciation for advanced economic thinking as well as the ability to frame interesting economic questions and to use economic analysis to evaluate them (Davis and Erekson, 1998). It is particularly in these courses that students should learn how economists do economics, so that the development of questions and how they are solved is emphasised.

Siegfried, Butler and Finegan (1998) point out that although the intermediate courses tend to effectively develop economic theory, they often do not stress the relevance of the theories by testing them with empirical data, using models to solve a variety of problems, and evaluating the differences between competing models. They further state that 'to achieve the overall objective of the major the intermediate macro and micro courses must emphasise active student learning, practice in applying what students learn, and the exercise of critical judgment (p. 68)'. Additionally, depth rather than breadth of subject matter should be stressed so that students are exposed to a variety of ways in which the same model can be used to solve problems of increasing difficulty. In order to stimulate learning, students must actively participate in a process in which a variety of instructional methods are used. Davis and Erekson (1998) suggest incorporating assignments with real-world data, requiring more complex research and paper assignments, asking interpretative questions that require critical analysis of data and using pedagogical methods which involve active and cooperative learning.

The role of technology in understanding mathematical concepts

The growing importance of technology in mathematics education

Because economic theory relies on mathematics as a foundation for economic reasoning, it is helpful to investigate the role of technology in mathematics education. During the 1980s, the need for change in pedagogical practices in the teaching of mathematics, science, and technology was recognised. Three U.S. documents (National Commission on Excellence in Education, 1983; National Science Board Commission on Precollege Education in Mathematics, Science, and Technology, 1983; National Research Council (NRC), 1989) were paramount in reforming the way mathematics and science was taught. The imperatives developed in these reports emphasised greater reliance on problem-solving approaches and the incorporation of more technology to assist in implementing these goals. The NRC further emphasised that increased use of technology within society will necessitate a greater development of mathematical ideas and students' ability to think critically and solve problems. The implication for mathematics education was a change in focus with an emphasis on developing the ability to apply skills and seek creative solutions to meaningful tasks rather than memorising and performing a specific set of skills.

In its *Principles and Standards for School Mathematics* (2000), the National Council of Teachers of Mathematics (NCTM) describes the standards that should permeate the mathematics curriculum. Underlying these standards are six principles which include *the technology principle*. The technology principle states that 'technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. As essential tools for teaching and learning, calculators and computers provide visual images of mathematical ideas, facilitate organising and analysing data, and compute efficiently and accurately. Further, when such tools are available, students can focus more on mathematical process such as decision-making, reasoning, and problem solving (p. 24)'.

During the past two decades, technology has changed mathematics education in three important ways (Van de Walle, 2001). First, some mathematics taught previously is now obsolete. Second, in many cases, we can teach mathematics more effectively with technology. Third, and probably most important, some topics that were previously inaccessible to undergraduate students with limited training in mathematics can now be made accessible to students in meaningful ways.

The effectiveness of technology as a pedagogical tool in mathematics

Many research efforts have examined the integration of technology into the teaching and learning of mathematics. These studies support the hypothesis that technology can enhance learning and they suggest ways in which pedagogical practices may be augmented in order to facilitate the use of technology. In summarising the impact of technology on mathematics education, Kaput (1992) discusses the pedagogical benefits derived from the use of calculators and computers in the mathematics classroom. These benefits include a richer problem-solving environment as more time can be devoted to conceptual understanding rather than skill development, the graphical advantage of computing utilities, in addition to the potential for a deeper understanding of algebraic ideas.

Multiple studies have provided empirical support for the infusion of technology into the mathematics classroom. These studies typically use quantitative methods to evaluate differences in learning mathematics by comparing a control group in a traditional classroom using lectures as the primary pedagogical practice to an experimental group in a technologically-intensive classroom using a computer algebra system. For example, three studies (Hawker, 1986; Heid, 1988; Judson, 1988) found evidence that students using computer algebra systems perform as well as or better than students in traditional business calculus courses. In a broader study, the performance of university students taught engineering calculus using a computer algebra system was compared to the performance of students using paper-and-pencil computations (Palmiter, 1991). Both groups received lecture as a primary means of instruction throughout the semester. However, after five weeks, the computer algebra system group also spent time engaged in active learning with computer-oriented labs. Empirical results indicated that students who used the computer algebra system scored higher on tests covering both their conceptual knowledge as well as their ability to do computations. In addition, student surveys at the end of the semester indicated that 85% of those in the computer algebra system section were confident in continuing in the calculus sequence compared to only 69% of the traditional group. Finally, 95% of the computer algebra system section indicated they would sign up for a course with computer algebra systems again and 43% indicated an improvement in attitude toward mathematics and computers.

O'Callaghan (1998) finds similar results in a study of students' conceptual knowledge of function when he compares students in a traditional college algebra class with students in an experimental Computer Intensive Algebra (CIA) class. He found that the CIA students achieved a better overall understanding of functions and were also better able to model, interpret and translate information. The CIA students also showed significant improvement in their attitude toward mathematics, were less anxious about mathematics and rated their class as more interesting. In addition, a higher percentage of students successfully completed the CIA course compared to the traditional lecture based group.

Mayes (1995) notes that software tools are typically employed as a pedagogical method to reduce computation and manipulation, visualise algebraic concepts by relating graphical and algebraic representations of those concepts, and focus on the utility of mathematics in modelling and solving application problems. He argues that research shows that students often fail to make the correct connections between the visual/graphical representation and the analytic/algebraic one. Like O'Callaghan (1998), Mayes (1995) compares a traditional lecture-based college algebra course to an experimental algebra course stressing active student involvement and the use of the computer. He found that the experimental group had significantly higher scores on inductive reasoning, visualisation and problem solving while demonstrating an equivalent level of manipulation and computational skills.

Additional benefits of technology may include more cooperation among students. Rather than making the teaching and learning episodes more solitary, some researchers argue that technology can enhance collaborative learning. Miller and Castellanos (1996) find that technology facilitates collaborative learning based on their research with high school juniors involved in a university summer programme. Corroborating prior assertions by the Center for Technology in Education Staff (1993), the authors suggest the following criteria on which to judge best practice in the technology-infused classroom: (i) higher level thinking is stressed, (ii) students help each other learn, and (iii) learning occurs across different domains and disciplines. These criteria are very closely connected to the characteristics of collaborative learning practices.

Maple as a pedagogical tool

Several research endeavours specifically discuss the computer algebra system *Maple* as a tool for enhancing pedagogical practices in an undergraduate calculus class (e.g. Boyce and Ecker, 1995; Hillel, 1992; Judson, 1990). For example, Boyce and Ecker (1995) find that the incorporation of *Maple* in a computer-intensive calculus class had a substantial impact both on the mode of instruction as well as course content, the latter being shifted toward visualisation, conceptual understanding, modelling, and higher-level problem solving. They also suggest that the use of *Maple* eliminates the necessity of framing class demonstrations and homework problems to teach students how to carry out hand computations. This allows classroom instructors to focus on the rationale for performing these tasks and on

the conclusions that can be drawn from results. It also enables more interesting applications to be considered, placing greater emphasis on how problems are modelled. Mathews' (1990) report of using *Maple* to teach Riemann sums illustrates the efficiency of *Maple* in obtaining more precise solutions, versus extended hand-computed calculations where students often lose sight of the goals of the exercise. Similarly, Judson (1991) found that using *Maple* technology in a calculus lab reinforced basic concepts, allowing students to achieve a deeper understanding and making the learning process more enjoyable. Enthusiasm was high, creating a pleasant, relaxed and spontaneous atmosphere. Students felt challenged, but most could work independently. Galbraith and Pemberton (2001) also found a strong correlation between confidence and motivation after integrating *Maple* into undergraduate mathematics courses in Australia. Positive attitudes as well as an expanded range of student-generated questions significantly increased when students learned with *Maple*.

In an analysis of technologies, Alkhateeb (2002) compared the achievement, attitudes toward success in mathematics and mathematics anxiety of college calculus students using *Maple* with students using a graphing calculator. Although his results showed no significant differences in achievement between the groups, students using the computer had a more positive attitude and experienced less anxiety. Like Boyce and Ecker's (1995) study, the use of *Maple* had a substantial influence on the mode of instruction and course content.

Finally, research suggests that *Maple* is among the best software packages to use in the mathematics classroom (Ponidi, 1999). In a comparison of numerous software packages used to solve partial differential equations problems, Ponini concludes that *Maple* is most effective in solving, comparing and visualising numerical and exact solutions. He notes an increase in student enthusiasm, more active involvement in group and individual assignments, as well as new perceptions about modelling, solving, visualising and interpreting the partial differential equations problems.

Conclusions

In summary, our paper suggests the importance of incorporating technologies such as *Maple* in the economics classroom to enhance teaching mathematical concepts. The benefits of this teaching practice are theoretically motivated by merging the current literature on active and cooperative learning in economics with extensive research on the efficacy of implementing technology in mathematics education. We find that both branches of research have similar implications for economics pedagogy: that is, active learning activities in general, and technology in particular,

enhance the ability of students to more effectively learn mathematics in economics. While most previous papers on the use of technology in economics provide examples of how technology can be used in economics pedagogy, the primary focus of our paper is to examine the rationale and conditions under which the implementation of this type of technology may enhance learning. The examples from intermediate economic theory are provided to illustrate how implications from education research in economics and mathematics might be applied in the classroom.

We close with some caveats for practitioners. As an active learning technique, the primary benefit of technology is that it enhances the ability of the student to transfer new material to a variety of situations so that analytical and critical thinking skills are fostered. However, opportunity costs exist due to the time and effort involved in learning how to use the new technology and in designing implementation strategies. Incorporating more technology often means that less time is available to fully cover other topics. As both our experience and Salemi (2002) suggests, this implies an efficient use of technology should focus on teaching those topics that are relatively important and difficult to master. Students are often anxious about the use of mathematics to convey economic logic. In addition, the prospect of learning a new technology in order to help them grasp certain mathematical concepts within economics can initially appear to be twice the trouble. However, it is our experience that although students may at first question the wisdom of such an approach, they come to see its value by the time the course is completed, provided certain pitfalls are avoided. [8]

Experience has shown that there are several conditions which maximise the effectiveness of using technology in order to facilitate the assimilation of mathematical concepts in economic theory. First, the instructor should intuitively explain the mathematical concept that is to be integrated. General examples that enhance conceptual understanding should be used rather than restricting the opening discussion to examples related to economics so that students see that the economics examples are part of a larger concept. This facilitates acquisition and retention of the new mathematical concept. For example, constrained maximisation involves contour lines such as those found on a topographical map with which most students are familiar. The notion that indifference curves and isoquants are conceptually identical to these contour lines can be introduced later once the students have connected the mathematical concept with something previously learned.

Second, as new material is explained, the examples must be simply stated, geared to familiarise the student with basic commands and code. Allow students to make small changes and re-execute the program.

Third, create assignments that allow students to work from a template, requiring that they make appropriate changes in key components of the model. This will facilitate an understanding of how the variables are interrelated as well as their impact on the overall model.

Fourth, use class time judiciously. The focus should move quickly to the economic problem being studied, and how the mathematics can be used to express the nature of the problem and lead to a solution. The technology is merely a conduit. The goal is not to create master programmers, but to allow students to use mathematics to solve a familiar economic problem. Remember that the motives for employing this technology in economics courses may differ from those in mathematics.

Finally, it is very helpful (though not necessary) if many of the students have already been exposed to the technology in courses in other disciplines such as mathematics, thereby connecting new information to previously learned material and increasing the ability for acquisition and retention to occur.

Using this procedure makes it possible to use technology such as *Maple* to illustrate and explain new and relatively complex mathematical applications is as little as two or three sessions in a computer classroom. If learning the technology becomes an obstacle, either because it requires extensive use of class time or because students do not possess sufficient skills, the technology ceases to be effective. In these cases, significant opportunity costs may prohibit its incorporation. However, if the technology is effectively applied, research in mathematics education as well as the research on active and cooperative learning strategies in economics is consistent in suggesting that it can be very effective as a pedagogical tool.

Again note that colour examples are available in the online appendix, http://faculty.bellarmine.edu/fraymond/teaching.asp. Although the authors hope that readers find the examples in the appendix useful, the primary contribution of this research is to answer why, when and to what extent this technology may be useful. Our findings suggest that economists should become more familiar with related research in other disciplines in order to gain additional insight concerning appropriate applications of technology in economics pedagogy.

Appendix

We provide eight examples from intermediate microeconomic and macroeconomic theory. The first example illustrates utility functions and indifference curves. The second illustrates consumer maximisation. The third represents the producer problem of cost minimisation. The last example demonstrates how *Maple* can be used to solve for Keynesian macroeconomic equilibrium. These examples are available in colour via the web, http://faculty.bellarmine.edu/fraymond/teaching.asp.

While students may grasp the concept of an indifference curve, they often have difficulty envisioning a single indifference curve as part of an indifference map. This first example can be used to familiarise the student with the basic syntax of *Maple* while depicting the indifference curve as a contour line defined by $\{x, y\}$ such that $f(x, y) = k$ where k is a real-valued constant. Note that in the following examples, *Maple* code is preceded by the '>' symbol. One may include comments by omitting this symbol.

Example 1

Graph of Cobb-Douglas Utility: The Indifference Map

```
> restart;
> u:=3*x^(2/3)*y^(1/3);
```
$$u := 3\, x^{(2/3)}\, y^{(1/3)}$$

```
> plot3d(u,x=0..100,y=0..100,labels=["x","y","utility"],title=
"Figure 1: Indifference Map",axes=boxed,thickness=4);
```

Figure 1: Indifference Map

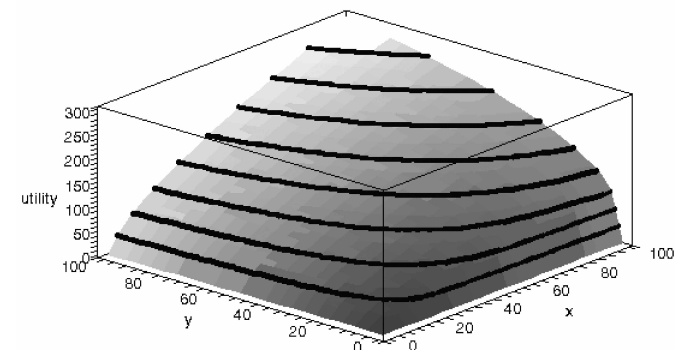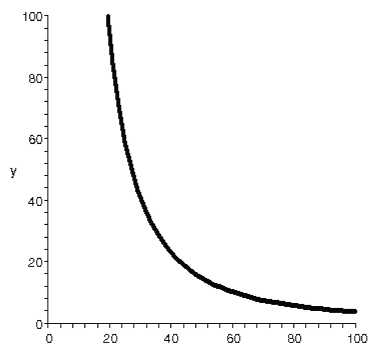**Figure 2:** Indifference Curve for 100 utils

Graphing Indifference Curves

```
> y1:=solve(u=100,y);
```

$$y1 := \frac{1000000}{27\, x^{\frac{3}{2}}}$$

```
> plot(y1,x=0..100,y=0..100,labels=["x","y"],title="Figure 2:
Indifference Curve for 100 utils",color=black,thickness=4);
```

The next example illustrates a solution to the consumer utility maximisation problem using Lagrange multipliers. The *Maple* syntax helps one to clearly distinguish the felicity (function to be optimised) from the budget constraint.

Example 2

Example of Consumer Optimisation with Quasi-linear Preferences

```
> restart;
```
Type in utility function.
```
> u:=y+x^(1/2);
```

$$u := y + \sqrt{x}$$

Type in the price of good x.
```
> p[x]:=5;
```

$$p_x := 5$$

Type in the price of good y.
```
> p[y]:=7;
```

$$p_y := 7$$

Type in the amount in the budget, M.
```
> M:=20;
```

$$M := 20$$

```
> Mo:=p[x]*x+p[y]*y-M;
```

$$Mo := 5\,x + 7\,y - 20$$

Set up the Lagrangian.
```
> L:=u-lambda*(Mo);
```

$$L := y + \sqrt{x} - \lambda\,(5\,x + 7\,y - 20)$$

Find the first order condition of the Lagrangian with respect to x.
```
> Diff(L,x);
```

$$\frac{\partial}{\partial x}(y + \sqrt{x} - \lambda\,(5\,x + 7\,y - 20))$$

```
> Lx:=value(%);
```

$$Lx := \frac{1}{2\,\sqrt{x}} - 5\,\lambda$$

Find the first order condition of the Lagrangian with respect to y.
```
> Diff(L,y);
```

$$\frac{\partial}{\partial y}(y + \sqrt{x} - \lambda\,(5\,x + 7\,y - 20))$$

```
> Ly:=value(%);
```

$$Ly := 1 - 7\,\lambda$$

The following solution (soln) is the optimal consumer bundle.
```
> soln:=solve({Lx=0,Ly=0,Mo=0},{x,y,lambda});
```

$$soln := \left\{ y = \frac{351}{140},\ \lambda = \frac{1}{7},\ x = \right.$$

```
> umx:=eval(u,soln);
```

$$umx := \frac{351}{140} + \frac{1}{100}\,\sqrt{49}\,\sqrt{100}$$

The following value is the maximum utility level, indicative of successful consumer optimisation.
```
> umax:=evalf(umx);
```

$$umax := 3.207142857$$

Next we specify the budget constraint by defining the budget constraint as a function of y in terms of x and income.
```
> xint:=M/p[x];
```

$$xint := 4$$

```
> yint:=M/p[y];
```

$$yint := \frac{20}{7}$$

```
> BC:=x->yint-(p[x]/p[y])*x;
```

$$BC := x \rightarrow yint - \frac{p_x \, x}{p_y}$$

```
> indy:=solve(u=umax,y);
```

$$indy := -1. \sqrt{x} + 3.207142857$$

```
> indcv:=x->indy;
```

$$indcv := x \rightarrow indy$$

The graph below illustrates the optimal solution to this consumer problem.

```
> plot([BC(x),indcv(x)],x=0..xint,labels=["good x","good
y"],title="Figure 3: Consumer Equilibrium",thickness=3);
```

Examine the utility function and the representative indifference curve depicted here. Can you describe the bias?

Figure 3 clearly illustrates the bias of the quasi-linear utility function towards good y.

Example 3 illustrates the similarities and differences between consumer and producer theory. The *Maple* syntax indicates that although they illustrate different economic concepts, clearly they are both problems involving constrained maximisation. The similarities and differences become more evident when the student is requested to make certain changes to the template provided by the instructor. In order to provide more of a challenge, the instructor may decide to use

Figure 3: Consumer Equilibrium
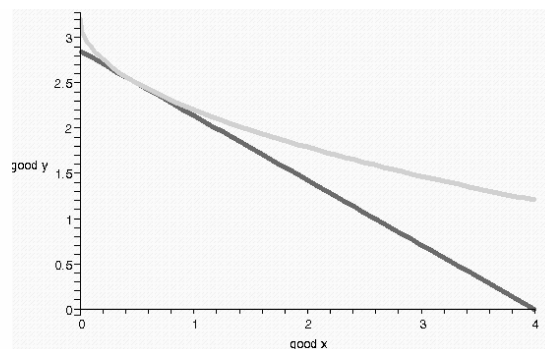
the consumer's utility maximisation problem as the template for the producer's cost minimisation problem. Note the slightly more general approach.

Example 3

This program will determine the Minimum Cost with Cobb-Douglas Production.

```
> restart;
> assume(L,nonnegative);
> assume(K,nonnegative);
> assume(lambda,nonnegative);
>
```

The Cobb-Douglas Production Function:

```
> F:=A*L^(delta)*K^(epsilon);
```

$$F := A \, L\text{\textasciitilde}^{\delta} \, K\text{\textasciitilde}^{\varepsilon}$$

```
> "Type in values for A, delta and epsilon.";
```

$$\text{"Type in values for A, delta and epsilon."}$$

```
> A:=1;
```

$$A := 1$$

```
> delta:=2/3;
```

$$\delta := \frac{2}{3}$$

```
> epsilon:=1/3;
```

$$\varepsilon := \frac{1}{3}$$

Type in the wage.

```
> w:=2;
```

$$w := 2$$

Type in the rental rate.

```
> r:=5;
```

$$r := 5$$

```
>
```

Type in the desired output level.

```
>
> q:=100;
```

$$q := 100$$

The Cost Function Follows.

```
> C:=r*K+w*L;
```
$$C := 5\,K\!\sim + 2\,L\!\sim$$

```
>
```
The output constraint follows:
Note: This becomes a minimisation problem when Qo<=0.
```
> Qo:=q-F;
```
$$Qo := 100 - L\!\sim^{(2/3)}\ K\!\sim^{(1/3)}$$

```
> Lag:=C+lambda*(Qo);
```
$$Lag := 5\,K\!\sim + 2\,L\!\sim + \lambda\!\sim \left(100 - L\!\sim^{(2/3)}\ K\!\sim^{(1/3)} \right)$$

```
> Diff(Lag,L);
```
$$\frac{\partial}{\partial L\!\sim} \left(5\,K\!\sim + 2\,L\!\sim + \lambda\!\sim \left(100 - L\!\sim^{(2/3)}\ K\!\sim^{(1/3)} \right) \right)$$

```
> LL:=value(%);
```
$$LL := 2 - \frac{2\,\lambda\!\sim\,K\!\sim^{(1/3)}}{3\,L\!\sim^{(1/3)}}$$

```
> Diff(Lag,K);
```
$$\frac{\partial}{\partial K\!\sim} \left(5\,K\!\sim + 2\,L\!\sim + \lambda\!\sim \left(100 - L\!\sim^{(2/3)}\ K\!\sim^{(1/3)} \right) \right)$$

```
> LK:=value(%);
```
$$LK := 5 - \frac{\lambda\!\sim\,L\!\sim^{(2/3)}}{3\,K\!\sim^{(2/3)}}$$

The following solution is the cost-minimising bundle.
```
>
> soln:=fsolve({LL=0,LK=0,Qo=0},{L,K,lambda});
```
$$soln := \{K\!\sim = 34.19951893,\ L\!\sim = 170.9975947,\ \lambda\!\sim = 5.129927840\}$$

```
>
```
The following value is the mimimum cost.
```
> minc:=eval(C,soln);
```
$$minc := 512.9927840$$

```
> lint:=minc/w;
```
$$lint := 256.4963920$$

```
> kint:=minc/r;
```
$$kint := 102.5985568$$

```
> TC:=L->kint-(w/r)*L;
```
$$TC := L \rightarrow kint - \frac{w\,L}{r}$$

```
>
> isoK:=solve(q=F,K);
```
$$isoK := \frac{1000000}{L\!\sim^{2}}$$

```
> isoquant:=L->isoK;
```
$$isoquant := L \rightarrow isoK$$

The graph below illustrates the optimal solution.
```
> plot([TC(L),isoquant(L)],L=0..lint,K=0..kint,labels=["Labor"
,"Capital"],title="Production Equilibrium",thickness=3);
```

To illustrate that this is indeed the minimum cost, we evaluate the cost of two nearby points of the same isoquant.
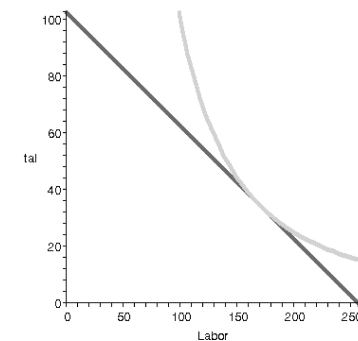```
> L1:=eval(L,soln)-.2;
```
$$L1 := 170.7975947$$

```
> L2:=eval(L,soln)+.2;
```
$$L2 := 171.1975947$$

**Figure 4:** Production Equilibrium

```
> K1:=eval(isoquant(L),L=L1);
```
$$KI := 34.27965949$$

```
> K2:=eval(isoquant(L),L=L2);
```
$$K2 := 34.11965905$$

```
> C1:=eval(w*L1+r*K1);
```
$$CI := 512.9934868$$

```
> C2:=eval(w*L2+r*K2);
```
$$C2 := 512.9934846$$

Recall that the minimum cost was 512.9927841 < C1 and C2.

In the next example we solve for and demonstrate the Hicksian substitution and income effects.

### Example 4

Hicks Substitution and Income Effects
```
> assume(x,nonnegative);
> assume(y,nonnegative);
> assume(lambda,nonnegative);
```

Type in values for A, delta and epsilon.
```
> A:=1;
```
$$A := 1$$

```
> delta:=1/2;
```
$$\delta := \frac{1}{2}$$

```
> epsilon:=1/2;
```
$$\varepsilon := \frac{1}{2}$$

The Cobb-Douglas Utility Function follows.
```
> u:=A*x^(delta)*y^(epsilon);
```
$$u := \sqrt{x\sim}\,\sqrt{y\sim}$$

Type in the price of good x. We will investigate the effect of a price increase in good x.
```
> p1[x]:=2;
```
$$pI_{x\sim} := 2$$

```
> p2[x]:=3;
```
$$p2_{x\sim} := 3$$

Type in the price of good y.

```
> p[y]:=50/8;
```
$$p_{y\sim} := \frac{25}{4}$$

Type in the budget amount.
```
> M:=100;
```
$$M := 100$$

Write the budget constraints, before and after the increase in the price of x.
```
> M1:=M-(p1[x]*x+p[y]*y);
```
$$MI := 100 - 2\,x\sim - \frac{25}{4}y\sim$$

```
> M2:=M-(p2[x]*x+p[y]*y);
```
$$M2 := 100 - 3\,x\sim - \frac{25}{4}y\sim$$

Determine the first Lagrangian equation.
```
> L1:=u+lambda[1]*(M1);
```
$$LI := \sqrt{x\sim}\,\sqrt{y\sim} + \lambda\sim_1\left(100 - 2\,x\sim - \frac{25}{4}y\sim\right)$$

```
> Diff(L1,x);
```
$$\frac{\partial}{\partial x\sim}\left(\sqrt{x\sim}\,\sqrt{y\sim} + \lambda\sim_1\left(100 - 2\,x\sim - \frac{25}{4}y\sim\right)\right)$$

```
> L1x:=value(%);
```
$$LIx := \frac{\sqrt{y\sim}}{2\sqrt{x\sim}} - 2\lambda\sim_1$$

```
> Diff(L1,y);
```
$$\frac{\partial}{\partial y\sim}\left(\sqrt{x\sim}\,\sqrt{y\sim} + \lambda\sim_1\left(100 - 2\,x\sim - \frac{25}{4}y\sim\right)\right)$$

```
> L1y:=value(%);
```
$$LIy := \frac{\sqrt{x\sim}}{2\sqrt{y\sim}} - \frac{25}{4}\lambda\sim_1$$

Next, determine the Lagrangian equation after the price of x rises.
```
> L2:=u+lambda[2]*(M2);
```

$$L2 := \sqrt{x\!\sim}\,\sqrt{y\!\sim} + \lambda\!\sim_2\left(100 - 3\,x\!\sim - \frac{25}{4}y\!\sim\right)$$

```
> Diff(L2,x);
```

$$\frac{\partial}{\partial x\!\sim}\left(\sqrt{x\!\sim}\,\sqrt{y\!\sim} + \lambda\!\sim_2\left(100 - 3\,x\!\sim - \frac{25}{4}y\!\sim\right)\right)$$

```
> L2x:=value(%);
```

$$L2x := \frac{\sqrt{y\!\sim}}{2\sqrt{x\!\sim}} - 3\,\lambda\!\sim_2$$

```
> Diff(L2,y);
```

$$\frac{\partial}{\partial y\!\sim}\left(\sqrt{x\!\sim}\,\sqrt{y\!\sim} + \lambda\!\sim_2\left(100 - 3\,x\!\sim - \frac{25}{4}y\!\sim\right)\right)$$

```
> L2y:=value(%);
```

$$L2y := \frac{\sqrt{x\!\sim}}{2\sqrt{y\!\sim}} - \frac{25}{4}\lambda\!\sim_2$$

The following solutions (soln1 and soln2) are the optimal bundles before and after the rise in the price of x.

$$soln1 := \left\{\lambda\!\sim_1 = RootOf(-1 + 50\,\_Z^2, 0.1414213562),\ x\!\sim = 25,\ y\!\sim = 8\right\}$$

```
> soln2:=solve({L2x=0,L2y=0,M2=0},{x,y,lambda[2]});
```

$$soln2 := \left\{\lambda\!\sim_2 = RootOf(-1 + 75\,\_Z^2, 0.1154700538),\ y\!\sim = 8,\ x\!\sim = \frac{50}{3}\right\}$$

```
> umx1:=eval(u,soln1);
```

$$umx1 := \sqrt{25}\,\sqrt{8}$$

```
> umx2:=eval(u,soln2);
```

$$umx2 := \frac{1}{3}\sqrt{50}\,\sqrt{3}\,\sqrt{8}$$

The values for umax1 and umax2 denote the optimal levels of utility before and after the price increase.

```
> umax1:=evalf(umx1);
```

$$umax1 := 14.14213562$$

```
> umax2:=evalf(umx2);
```

$$umax2 := 11.54700539$$

Here are the x and y intercepts for the two budget constraints.

```
> x1int:=round(M/p1[x]);
```

$$x1int := 50$$

```
> x2int:=round(M/p2[x]);
```

$$x2int := 33$$

```
> yint:=M/p[y];
```

$$yint := 16$$

Arange in functional form for the purpose of graphing the budget constraints.

```
> BC1:=x->yint-(p1[x]/p[y])*x;
```

$$BC1 := x \rightarrow yint - \frac{p1_x\,x}{p_y}$$

```
> BC2:=x->yint-(p2[x]/p[y])*x;
```

$$BC2 := x \rightarrow yint - \frac{p2_x\,x}{p_y}$$

Solve and graph the two indifference curves indicating utility before and after the price increase.

```
> indy1:=solve(u=umax1,y);
```

$$indy1 := \frac{199.9999999}{x\!\sim}$$

```
> indy2:=solve(u=umax2,y);
```

$$indy2 := \frac{133.3333335}{x\!\sim}$$

```
> indcv1:=x->indy1;
```

$$indcv1 := x \rightarrow indy1$$

```
> indcv2:=x->indy2;
```
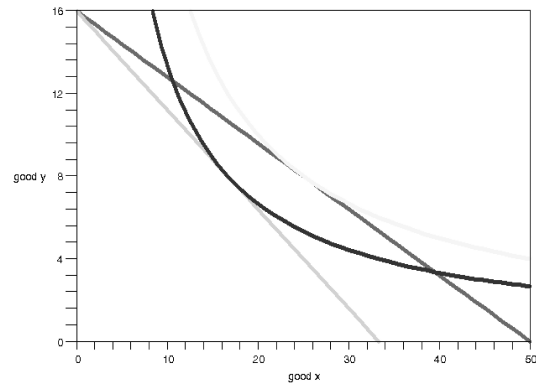
$$indcv2 := x \rightarrow indy2$$

The graph below illustrates the optimal solutions before (red and yellow tangency) and after (green and blue tangency) the price of x increases.

```
>
plot([BC1(x),BC2(x),indcv1(x),indcv2(x)],x=0..x1int,y=0..yint,
labels=["good x","good y"],title="Figure 5: Consumer
Equilibrium",thickness=4,axes=boxed);
```

Figure 5: Consumer Equilibrium

To identify the Hicks substitution and income effects for x, look at the choice the consumer would make if allowed to maintain original utility level at the new price ratio.

```
> pr:=p2[x]/p[y];
```

$$pr := \frac{12}{25}$$

```
> u1:=umax1-u;
```

$$u1 := 14.14213562 - \sqrt{x\sim}\sqrt{y\sim}$$

```
> u1y:=solve(u1=0,y);
```

$$u1y := \frac{199.9999999}{x\sim}$$

Find the marginal rate of substitution.

```
> Diff(u1y,x);
```

$$\frac{d}{dx\sim}\left(\frac{199.9999999}{x\sim}\right)$$

```
> u1slope:=-value(%);
```

$$u1slope := \frac{199.9999999}{x\sim^2}$$

Use the fact that at the point of tangency, the price ratio equals the marginal rate of substitution in order to find the x and y values associated with the Hicksian solution.

```
> Hicks:=solve(pr=u1slope,x);
```

$$Hicks := 20.41241452, -20.41241452$$

```
> Hicksx:=max(Hicks);
```

$$Hicksx := 20.41241452$$

```
> Hicksy:=eval(u1y,x=Hicksx);
```

$$Hicksy := 9.797958967$$

Numerically solve for the Hicks substitution and income effects.

```
> x1:=solve(soln1[3]);
```

$$x1 := 8$$

```
> SubEffect:=x1-Hicksx;
```

$$SubEffect := -12.41241452$$

```
> x2:=solve(soln2[3]);
```

$$x2 := \frac{50}{3}$$

```
> IncEffect:=Hicksx-x2;
```

$$IncEffect := 3.74574785$$

Next, rearrange format for graphing the Hicksian budget constraint.
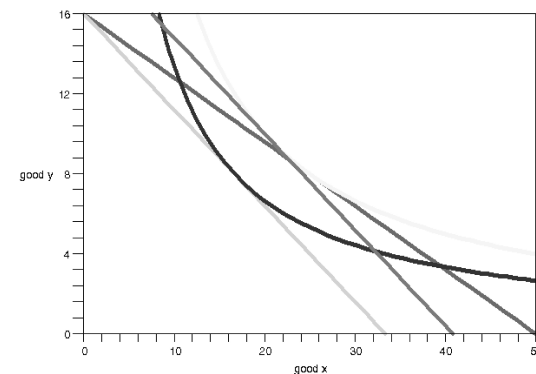
```
> Mg:=p2[x]*Hicksx+p[y]*Hicksy;
```

$$Mg := 122.4744871$$

```
> BC3:=x->(Mg/p[y])-(p2[x]/p[y])*x;
```

$$BC3 := x \rightarrow \frac{Mg}{p_y} - \frac{p2_x\,x}{p_y}$$

The graph below indicates the Hicks substitution and income effects

Figure 6: Hicks Substitution and Income Effects for Good x

```
>
plot([BC1(x),BC2(x),indcv1(x),indcv2(x),BC3(x)],x=0..x1int,y=0
..yint,labels=["good x","good y"],title="Figure 6: Hicks
Substitution and Income Effects for Good
x",thickness=4,axes=boxed);
```

Questions

On the graph:
1) identify the original optimal bundle,
2) identify the optimal bundle after the price change,
3) identify the change in good x that describes the substitution effect,
4) identify the change in good x that describes the income effect.
5) Repeat (3) and (4) for good y.

The next four examples come from intermediate macroeconomic theory. The first example depicts the Keynesian Cross. The second solves the Keynesian IS-LM model. The final two examples illustrate the Solow model of economic growth, including the Golden Rule.

Example 5

The Keynesian Cross
```
> restart;
> assume(Y1>=0,TAX>=0,INV>=0,GOV>=0,IM>=0);
```
This program allows us to look at nonlinear versions of the model.
```
>
```
Government Expenditures and the Marginal Tax Rate
```
>
> GOV:=1000-0.01*Y;
```
$$GOV := 1000 - 0.01\ Y$$

```
> TAX:=0.05*Y;
```
$$TAX := 0.05\ Y$$

```
>
```
The Consumption Function
```
>
> CONS:=3000+15*(Y-TAX)^0.5;
```
$$CONS := 3000 + 14.62019152\ Y^{0.5}$$

The Investment Schedule
```
> INV:=1000+0.1*Y;
```

$$INV := 1000 + 0.1\ Y$$

```
>
```
Exogenous Exports
```
> EX:=600;
```
$$EX := 600$$

```
>
```
Imports
```
>
> IM:=400+0.1*(Y-TAX);
```
$$IM := 400. + 0.095\ Y$$

Define the Keynesian Cross
```
> AGGEXP:=CONS+GOV+INV+EX-IM;
```
$$AGGEXP := 5200. + 14.62019152\ Y^{0.5} - 0.005\ Y$$

```
>
```
Equilibrium GDP for this Model
```
> GDP:=solve({AGGEXP=Y});
```
$$GDP := \{Y = 6331.699000\}$$

Define Savings
```
> SAV:=INV+GOV+EX-IM-TAX;
```
$$SAV := 2200. - 0.055\ Y$$

Define Disposable Income
```
> DISPINC:=CONS+SAV;
```
$$DISPINC := 5200. + 14.62019152\ Y^{0.5} - 0.055\ Y$$

Evaluate these parameters with respect to equilibrium GDP.
```
> eval([Y,SAV,DISPINC],GDP);
```
$$[6331.699000, 1851.756555, 6015.114050]$$

```
> eval([TAX,INV,GOV,IM,EX],GDP);
```
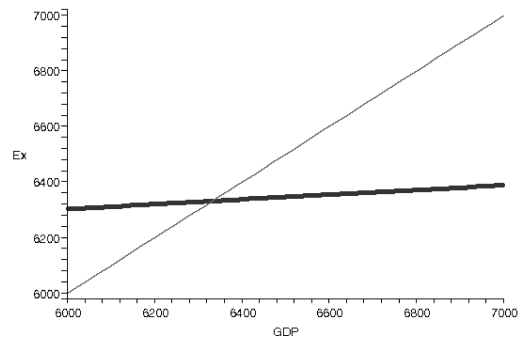$$[316.5849500, 1633.169900, 936.6830100, 1001.511\cdot$$

```
> p1:=plot([AGGEXP],Y=6000..7000,labels=["GDP","Ex"],title=
"Figure 7, Aggregate Expenditures",thickness=4,color=blue):
> f:=Y->Y;
```
$$f := Y \rightarrow Y$$

```
> p2:=plot([f(Y)],Y=6000..7000,thickness=2):
> with(plots):
```
Warning, the name changecoords has been redefined
```
> display(p1,p2);
```

**Figure 7:** Aggregate Expenditures



*Example 6*

The Keynesian IS-LM Model
```
> restart;
```
Government Expenditures and the Marginal Tax Rate
```
> GOV:=10000;
```
$GOV := 10000$
```
> TAX:=0.05*Y;
```
$TAX := 0.05\ Y$

The Consumption Function
```
> CONS:=8000+.85*(Y-TAX);
```
$CONS := 8000. + 0.8075\ Y$

The Investment Schedule
```
> INV:=2000-50*r;
```
$INV := 2000 - 50\ r$

Exports
```
> EX:=6000;
```
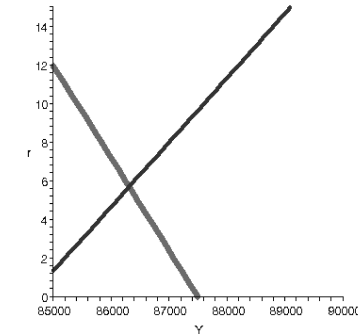$EX := 6000$

Imports
```
> IM:=5000+0.05*(Y-TAX);
```
$IM := 5000. + 0.0475\ Y$

Determine the Investment-Savings (IS) Function.
```
> AGGEXP:=CONS+GOV+INV+EX-IM;
```
$AGGEXP := 21000. + 0.7600\ Y - 50\ r$
```
> IS:=solve(Y=AGGEXP,r);
```

**Figure 8:** The Keyesian IS-LM Curves



$IS := -0.004800000000\ Y + 420.$
```
> p1:=plot(IS, Y=85000..90000,r=0..15,thickness=5):
>
>
```
Given the Money Supply.
```
> MS:=8500;
```
$MS := 8500$

Given Money Demand.
```
> MD:=40+0.1*Y-30*r;
```
$MD := 40 + 0.1\ Y - 30\ r$

Solve for Liquidity-Money (LM) Curve.
```
> LM:=solve(MS=MD,r);
```
$LM := -282. + 0.003333333333\ Y$
```
>
> p2:=plot(LM, Y=85000..90000,r=0..15,title="Figure 8: The
Keyesian IS-LM Curves",thickness=4,color=blue):
> with(plots):

> display(p1,p2,labels=["Y","r"]);
```

Solve for the Equilibrium GDP and Interest Rate.
Equilibrium GDP:
```
> eY:=solve(IS=LM,{Y});
```
$eY := \{Y = 86311.47541\}$

Equilibrium Interest Rate:
```
> er:=eval(LM,eY);
```
$er := 5.7049180$

*Example 7*

The Solow Growth Model

Though not essential for this application, the purpose of the inactive code (in black) is to provide related information and to more clearly reveal how the model is constructed.

```
> restart;
```
1) Define a CRS Production Function.
```
F:=(K,L)->A*K^.5*L^.5;
Diff(F(K,L),L);
MPL:=value(%);
Diff(F(K,L),K);
MPK:=value(%);
> A:=3;
```
$$A := 3$$

```
k:=K/L;
y:=Y/L;
```
Redefine Production in terms of Relative Capital.
```
> f:=k->A*k^.5;
```
$$f := k \rightarrow A\,k^{0.5}$$

2) Incorporate Investment, Consumption and Savings.
```
Y:=C+S;
C:=(1-sigma)*Y;
c:=C/L;
s:=S/L;
i:=I/L;
```
Leakages = Injections in Equilibrium.
```
s:=i;
```
Let sigma represent the marginal propensity to save. Then, 1-sigma is the marginal propensity to consume.
```
c:=(1-sigma)*y;
c:=y-s;
i=sigma*y;
> sigma:=.15;
```
$$\sigma := 0.15$$

3) The Capital Constraint
```
ChangeK:=I-delta*k;
> delta:=.10;
```

$$\delta := 0.10$$
```
> changek:= sigma*f(k)-delta*k;
```
$$changek := 0.45\,k^{0.5} - 0.10\,k$$

changek is the change in effective (K/L) capital.
4) Consumption
```
> cc:=f(k)-changek-delta*k;
```
$$cc := 2.55\,k^{0.5}$$

cc is consumption.
```
> p1:=plot(f(k),k=1..1000,title="Figure 9: Production and
Consumption",color=blue,thickness=3):
> p2:=plot(delta*k,k=1..1000, thickness=4):
> p3:=plot(cc,k=1..1000, color=brown,thickness=5):
```
Production, f(k), is in blue. Consumption is in brown. Depreciation is in red.
```
> with(plots):
> display(p1,p2,p3);
```

The final example is an extension of Example 7. It depicts the Golden Rule for capital accumulation. We derive the steady state level of capital necessary to maximise per capita consumption over time.

*Example 8*

The Golden Rule: Determining Optimal Steady State Consumption from the Solow Growth Model
```
> restart;
> A:=4;
```
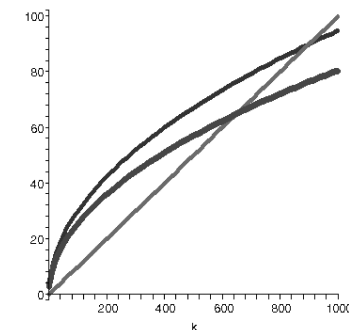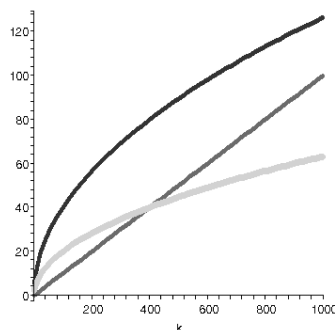$$A := 4$$

**Figure 9:** Production and Consumption

**Figure 10:** Golden Rule Production



```
> f:=k->A*k^.5;
```

$$f := k \rightarrow A\,k^{0.5}$$

```
>
> delta:=.10;
```

$$\delta := 0.10$$

```
> Diff(f(k),k);
```

$$\frac{d}{dk}(4\,k^{0.5})$$

```
> MPk:=value(%);
```

$$MPk := \frac{2.0}{k^{0.5}}$$

```
> grule:=solve(MPk=delta,k);
```

$$grule := 400.$$

The Golden Rule: grule is the optimal steady state level of capital associated with the highest possible sustainable level of consumption. It is given by the intersection of the savings (green) and depreciation (red) curves below.

```
> sigma:=delta*grule/f(grule);
```

$$\sigma := 0.5000000000$$

```
> Sigma is the optimal savings rate that guarantees a maximum
steady state stream of consumption.
> p1:=plot(f(k),k=1..1000,title="Figure 10: Golden Rule
Production",color=blue,thickness=3):
> p2:=plot(delta*k,k=1..1000,thickness=4):
> p3:=plot(sigma*f(k),k=1..1000,color=green,thickness=5):
```

```
> with(plots):
> Production, f(k), is in blue. Depreciation is red. Savings
is green.
> display(p1,p2,p3);
```

Notes

[1] The examples were originally generated using Waterloo Maplesoft's MAPLE version 9, 2003. Subsequent versions yield comparable output for these examples.

[2] The companion sites for these CAS packages are http://www.sciface.com, http://www.maplesoft.com, and http://www.wolfram.com.

[3] Our experience was consistent with this principle when MAPLE technology was incorporated into the intermediate microeconomics course. Previous exposure to MAPLE in earlier mathematics courses facilitated learning of the economic concepts. Using the technology to focus on a specific concept also made it easier to be more precise about what students were supposed to learn.

[4] Incorporating MAPLE assignments for specific problems simultaneously with the theoretical concepts on which they were based throughout the semester allowed us more easily to facilitate transferability as students 'practised' economics. As we discuss below, other benefits of using active learning techniques in class include immediate feedback to the instructor and students, an increase in collaborative learning, as well as opportunities for students to revise their thinking, etc.

[5] Cameron's (1998) review of the literature suggests that active learning *involves providing opportunities for students to meaningfully talk and listen, write, read, and reflect on the content, ideas, issues, and concerns of an academic subject (Meyers and Jones, 1993, p. xi). Learning is not a spectator sport. Students do not learn much just sitting in class listening to teachers, memorizing pre-packaged assignments, and spitting out answers. They must talk about what they are learning, write about it, relate it to past experiences, and apply it to their daily lives. They must make what they learn part of themselves (Bonwell, 1992, p.1)*

[6] Cameron (1998) maintains that cooperative learning requires students *to practice, at a higher level, positive interaction and individual accountability, as well as more sophisticated group-processing skills…groups work together over a longer period, such as several weeks or an entire semester, toward a shared goal (Meyers and Jones, 1993, p. 75).* She points out that assignments focus on problems that are too complicated or time consuming for one person to complete during the period over which the assignment is to be done.

[7] Salemi's comments are based on his discussions with other teachers and his experiences. His outline provides a succinct summary of why students learn more effectively when active learning strategies are employed to 'do economics'. In a separate paper (2005), Salemi argues that in-class active learning assignments are superior to out-of-class exercises because they give students an opportunity to explain their thinking, critically evaluate the thinking of their peers, obtain feedback, and revise their thinking (p. 49).

[8] As Salemi (2002) points out, active learning techniques may not be best for all students, as some students learn better by listening and then reflecting on lecture material. However, he argues that the existence of different learning styles is a reason for more, rather than less diversification.

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The Production of Mathematical Problems: a Diminishing Marginal Returns Experiment*

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Abstract

This article presents a classroom experiment to demonstrate several important production concepts, particularly the critical concept of the *diminish marginal returns to an input*. Although this experimental design shares principles with other variants of diminishing returns experiments described previously in the literature, it differs from them in two important feature: (1) it is specifically designed for large enrollment courses, and (2) it introduces the notion of capital as part of the experiment.

Playing in teams, students recreate a production process where they allocate some scarce resources (namely time, mathematical problems, students and calculators) to the production of mathematical solutions. Each round of production is allowed to change only by increasing labour (students) in marginal amounts while holding capital (calculators) constant. All teams are facing diminishing marginal returns to labour once the game is played for three or four rounds. Not only is this experience useful to introduce students to the nature of the critical concept of diminishing marginal returns, but with very little effort the instructor can expand the experiment to include dynamics related to issues of costs and profits.

Introduction

While students are usually excited to move from the abstract world of the *theory of the consumer* to the more concrete world of the *theory of production*, they usually have difficulties understanding the nature of the relationships in this latter area. For instance, many of them have problems conceptualising the fact that the