Optimal Macroeconomic Growth

P. Cartigny
GREQAM Université Méditerranée, Marseille France

St Andrews
Feb. 26 2009

Plan

1. Monosectorial optimal growth model.

Plan

1. Monosectorial optimal growth model.

   Ramsey, ...

Plan

1. Monosectorial optimal growth model.

   Ramsey, ...

2. Introduction to Optimal control theory, Calculus of variations.

Plan

1. Monosectorial optimal growth model.

   Ramsey, ...

2. Introduction to Optimal control theory, Calculus of variations.

Infinite horizon.

Plan

1. Monosectorial optimal growth model.

   Ramsey, ...

2. Introduction to Optimal control theory, Calculus of variations.

Infinite horizon.

Turnpike, Saddle point property.

Capital accumulation

Capital accumulation

- $k(t) \in R$, capital stock at time $t \in [0, \infty)$ (per capita)

Capital accumulation

- $k(t) \in R$, capital stock at time $t \in [0, \infty)$ (per capita)
- accumulation equation :

$$\dot{k}(t) = f(k(t)) - \alpha k(t) - c(t)$$

α (depreciation coeff.), $c(t)$ consumption.

Capital accumulation

- $k(t) \in R$, capital stock at time $t \in [0, \infty)$ (per capita)
- accumulation equation :

$$\dot{k}(t) = f(k(t)) - \alpha k(t) - c(t)$$

α (depreciation coeff.), $c(t)$ consumption.

- Assumptions :
  - $f(.)$: C^2, increasing,
  - $f''(.) < 0$,
  - $\lim_{k \to 0} f'(k) = +\infty$, $\lim_{k \to \infty} f'(k) = 0$

Welfare of the agents

Welfare of the agents

- Utility function $U(.)$

Welfare of the agents

- Utility function $U(.)$
- Assumptions :
  - C^2, $U'(.) > 0,\ U''(.) < 0$,

Welfare of the agents

- Utility function $U(.)$
- Assumptions :
  - C^2, $U'(.) > 0,\ U''(.) < 0$,
  - $\lim_{c \to 0} U'(c) = +\infty$

Welfare of the agents

- Utility function $U(.)$
- Assumptions :
  - C^2, $U'(.) > 0,\ U''(.) < 0$,
  - $\lim_{c \to 0} U'(c) = +\infty$
- Intertemporal actualised utility :

$$\int_0^{\infty} e^{-\delta t} U(c(t))\, dt$$

Welfare of the agents

- Utility function $U(.)$
- Assumptions :
  - C^2, $U'(.) > 0,\ U''(.) < 0$,
  - $\lim_{c \to 0} U'(c) = +\infty$
- Intertemporal actualised utility :

$$\int_0^{\infty} e^{-\delta t} U(c(t))\, dt$$

- Manager program :

$$\max_{c(.)} \quad \int_0^{\infty} e^{-\delta t} U(c(t)) dt$$
$$\dot{k}(t) = f(k(t)) - \alpha k(t) - c(t)$$
$$k_0$$

Welfare of the agents

- Utility function $U(.)$
- Assumptions :
  - C^2, $U'(.) > 0, \; U''(.) < 0$,
  - $\lim_{c \to 0} U'(c) = +\infty$
- Intertemporal actualised utility :

$$\int_0^{\infty} e^{-\delta t} U(c(t))\, dt$$

- Manager program :

$$\max_{c(.)} \quad \int_0^{\infty} e^{-\delta t} U(c(t)) dt$$
$$\dot{k}(t) = f(k(t)) - \alpha k(t) - c(t)$$
$$k_0$$

- Optimal Control Theory :Pontryagin (1950), Halkin (1975) Maximum Principle.

Infinite Horizon Calculus of variations Problem

Infinite Horizon Calculus of variations Problem

- $\mathcal{L}(k(t), \dot{k}(t)) := U(f(k(t)) - \alpha k(t) - \dot{k}(t))$

  On the eigenvalues associated to a saddle point.
- Interest of the Calculus of variations

Infinite Horizon Calculus of variations Problem

- $\mathcal{L}(k(t), \dot{k}(t)) := U(f(k(t)) - \alpha k(t) - \dot{k}(t))$

  On the eigenvalues associated to a saddle point.
- Interest of the Calculus of variations
- $J[k(.)] := \int_0^\infty e^{-\delta t} \mathcal{L}(k(t), \dot{k}(t)) dt$

Infinite Horizon Calculus of variations Problem

- $\mathcal{L}(k(t), \dot{k}(t)) := U(f(k(t)) - \alpha k(t) - \dot{k}(t))$

  On the eigenvalues associated to a saddle point.
- Interest of the Calculus of variations
- $J[k(.)] := \int_0^\infty e^{-\delta t} \mathcal{L}(k(t), \dot{k}(t)) dt$
  - $\mathcal{C}_{k_0} = \{k(.) \in BC^1([0, \infty)), k(0) = k_0\}$

Infinite Horizon Calculus of variations Problem

- $\mathcal{L}(k(t), \dot{k}(t)) := U(f(k(t)) - \alpha k(t) - \dot{k}(t))$

  On the eigenvalues associated to a saddle point.
- Interest of the Calculus of variations
- $J[k(.)] := \int_0^\infty e^{-\delta t}\mathcal{L}(k(t), \dot{k}(t))dt$
  - $\mathcal{C}_{k_0} = \{k(.) \in BC^1([0, \infty)), k(0) = k_0\}$

$$\max_{k(.)\in\mathcal{C}_{k_0}} \int_0^\infty e^{-\delta t}\mathcal{L}(k(t), \dot{k}(t))dt$$

Results

Results

Proposition 1 :

The functionnal $J[.]$ is well defined on BC^1, concave and C^1.

Results

Proposition 1 :
The functionnal $J[.]$ is well defined on BC^1, concave and C^1.

Proof

Results

Proposition 1 :
The functionnal $J[.]$ is well defined on BC^1, concave and C^1.

Proof

- $(u, v) \to \mathcal{L}(u, v) = U(f(u) - \alpha u - v)$ concave

Results

Proposition 1 :
The functionnal $J[.]$ is well defined on BC^1, concave and C^1.

Proof

- $(u, v) \to \mathcal{L}(u, v) = U(f(u) - \alpha u - v)$ concave
- Splitting of J :

Results

Proposition 1 :
The functionnal $J[.]$ is well defined on BC^1, concave and C^1.

Proof

- $(u, v) \to \mathcal{L}(u, v) = U(f(u) - \alpha u - v)$ concave
- Splitting of J :

1. $$\begin{array}{llll} \mathcal{T}: & BC^1(R_+, R) & \longrightarrow & BC^0(R_+, RxR) \\ & x(.) & \to & \mathcal{T}(x(.)) := (x(.), \dot{x}(.)) \end{array}$$

2. $$\begin{array}{llll} \mathcal{N}_\mathcal{L}: & BC^0(R_+, RxR) & \longrightarrow & BC^0(R_+, R) \\ & (x(.), y(.)) & \to & \mathcal{N}_\mathcal{L}(x(.), y(.)) := \mathcal{L}(x(.), y(.)) \end{array}$$

3. $$\begin{array}{llll} \mathcal{I}: & BC^0(R_+, R) & \longrightarrow & R \\ & u(.) & \to & \mathcal{I}(u(.)) := \int_0^\infty e^{-\delta t} u(t) dt \end{array}$$

Proposition 2 :

For $k(.) \in \mathcal{C}_{k_0}$, equivalence between :

Proposition 2 :

For $k(.) \in \mathcal{C}_{k_0}$, equivalence between :

1. $k(.)$ maximise $J[.]$ on $\mathcal{C}_{k_0}$

Proposition 2 :

For $k(.) \in \mathcal{C}_{k_0}$, equivalence between :

1. $k(.)$ maximise $J[.]$ on $\mathcal{C}_{k_0}$
2. $k(.)$ is solution on R_+ of Euler-Lagrange equation

$$\mathcal{L}_k(k(t), \dot{k}(t)) - \frac{d}{dt}\mathcal{L}_{\dot{k}}(k(t), \dot{k}(t)) + \delta\mathcal{L}_{\dot{k}}(k(t), \dot{k}(t)) = 0$$
$$k(0) = k_0$$

$$\ddot{k}(t) \quad +(\alpha - f'(k(t)))\,\dot{k}(t) + \frac{U'\{f(k(t)) - \alpha k(t) - \dot{k}(t)\}}{U''\{f(k(t)) - \alpha k(t) - \dot{k}(t)\}}\,\{\delta + \alpha - f'(k(t))\} \;=\; 0$$

$$\ddot{k}(t) \quad +(\alpha - f'(k(t)))\,\dot{k}(t) + \frac{U'\{f(k(t)) - \alpha k(t) - \dot{k}(t)\}}{U''\{f(k(t)) - \alpha k(t) - \dot{k}(t)\}}\,\{\delta + \alpha - f'(k(t))\} \;=\; 0$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\begin{aligned}\ddot{k}(t) \quad & +(\alpha - f'(k(t)))\,\dot{k}(t) + \\ & \frac{U'\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}{U''\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}\,\{\delta+\alpha-f'(k(t))\} \;=\; 0\end{aligned}$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\left\{ \begin{array}{l} \dot{h}_1(t) \\ \dot{h}_2(t) \end{array} \right\} = \left\{ \begin{array}{l} h_2(t) \\ (f'(h_1) - \alpha)h_2 - \frac{U'(\ldots)}{U''(\ldots)}(\delta + \alpha - f'(h_1)) \end{array} \right\}$$

$$\ddot{k}(t) \quad +(\alpha - f'(k(t)))\,\dot{k}(t) + \frac{U'\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}{U''\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}\,\{\delta+\alpha-f'(k(t))\} \;=\; 0$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\left\{ \begin{array}{l} \dot{h}_1(t) \\ \dot{h}_2(t) \end{array} \right\} = \left\{ \begin{array}{l} h_2(t) \\ (f'(h_1) - \alpha)h_2 - \frac{U'(...)}{U''(...)}(\delta + \alpha - f'(h_1)) \end{array} \right\}$$

Dynamical system :

$$\ddot{k}(t) \quad +(\alpha - f'(k(t)))\,\dot{k}(t) + \frac{U'\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}{U''\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}\{\delta+\alpha-f'(k(t))\} = 0$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\left\{ \begin{array}{l} \dot{h}_1(t) \\ \dot{h}_2(t) \end{array} \right\} = \left\{ \begin{array}{l} h_2(t) \\ (f'(h_1)-\alpha)h_2 - \frac{U'(...)}{U''(...)}(\delta+\alpha-f'(h_1)) \end{array} \right\}$$

Dynamical system :

- stationary solution (h_1^*, h_2^*)

$$\ddot{k}(t) \quad +(\alpha - f'(k(t)))\,\dot{k}(t) + \frac{U'\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}{U''\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}\,\{\delta+\alpha-f'(k(t))\} \;=\; 0$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\left\{\begin{array}{c} \dot{h}_1(t) \\ \dot{h}_2(t) \end{array}\right\} = \left\{\begin{array}{l} h_2(t) \\ (f'(h_1) - \alpha)h_2 - \frac{U'(...)}{U''(...)}(\delta + \alpha - f'(h_1)) \end{array}\right\}$$

Dynamical system :

- stationary solution (h_1^*, h_2^*)
- linearisation at (h_1^*, h_2^*)

$$\begin{aligned} \ddot{k}(t) & +(\alpha - f'(k(t)))\,\dot{k}(t) + \\ & \frac{U'\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}{U''\{f(k(t))-\alpha k(t)-\dot{k}(t)\}}\,\{\delta + \alpha - f'(k(t))\} = 0 \end{aligned}$$

let $h_1 = k \quad h_2 = \dot{k}$

$$\left\{ \begin{array}{l} \dot{h}_1(t) \\ \dot{h}_2(t) \end{array} \right\} = \left\{ \begin{array}{l} h_2(t) \\ (f'(h_1) - \alpha)h_2 - \frac{U'(...)}{U''(...)}(\delta + \alpha - f'(h_1)) \end{array} \right\}$$

Dynamical system :

- stationary solution (h_1^*, h_2^*)
- linearisation at (h_1^*, h_2^*)
- theorems : Hartman-Grobman, stable manifold

Stationary solution (h_1^*, h_2^*) such that

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0$$

Stationary solution (h_1^*, h_2^*) such that

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0$$

Let $h_1 = h_1^* + x_1$ and $h_2 = x_2$, linearisation at (h_1^*, h_2^*)

Stationary solution (h_1^*, h_2^*) such that

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0$$

Let $h_1 = h_1^* + x_1$ and $h_2 = x_2$, linearisation at (h_1^*, h_2^*)

$$\left\{ \begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array} \right\} = \left\{ \begin{array}{cc} 0 & 1 \\ A & \delta \end{array} \right\} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$$

where $A = \frac{U'\{f(h_1^*) - \alpha h_1^*\}}{U''\{f(h_1^*) - \alpha h_1^*\}} f''(h_1^*)$

Stationary solution (h_1^*, h_2^*) such that

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0$$

Let $h_1 = h_1^* + x_1$ and $h_2 = x_2$, linearisation at (h_1^*, h_2^*)

$$\left\{ \begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array} \right\} = \left\{ \begin{array}{cc} 0 & 1 \\ A & \delta \end{array} \right\} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$$

where $A = \frac{U'\{f(h_1^*)-\alpha h_1^*\}}{U''\{f(h_1^*)-\alpha h_1^*\}} f''(h_1^*)$
two eigenvalues

$$\lambda_\pm = \frac{1}{2}[\delta \pm \sqrt{\delta^2 + 4A}]$$

Stationary solution (h_1^*, h_2^*) such that

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0$$

Let $h_1 = h_1^* + x_1$ and $h_2 = x_2$, linearisation at (h_1^*, h_2^*)

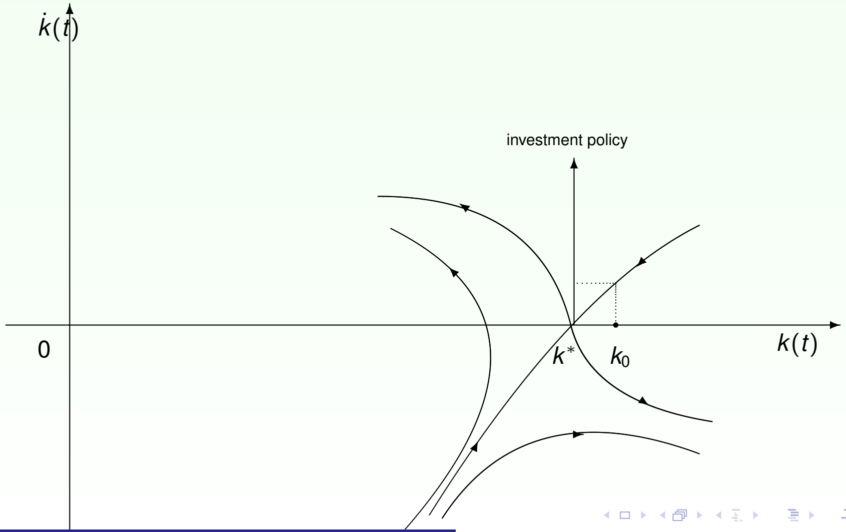
$$\left\{ \begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array} \right\} = \left\{ \begin{array}{cc} 0 & 1 \\ A & \delta \end{array} \right\} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$$

where $A = \frac{U'\{f(h_1^*)-\alpha h_1^*\}}{U''\{f(h_1^*)-\alpha h_1^*\}} f''(h_1^*)$
two eigenvalues

$$\lambda_\pm \;=\; \frac{1}{2}[\delta \pm \sqrt{\delta^2 + 4A}]$$

(h_1^*, h_2^*) saddle point

Stable Point Stability



Turnpike

Theorem

There exists a unique stationary optimal solution $k^* := (h_1^*, h_2^*)$

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0.$$

For any k_0 near k^*, there exists an optimal solution $k(.)$ that converges to k^* $(t \to +\infty)$.

Turnpike

Theorem
There exists a unique stationary optimal solution $k^* := (h_1^*, h_2^*)$

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0.$$

For any k_0 near k^*, there exists an optimal solution $k(.)$ that converges to k^* ($t \to +\infty$).

It is the only solution belonging to BC^1

Turnpike

Theorem
There exists a unique stationary optimal solution $k^* := (h_1^*, h_2^*)$

$$f'(h_1^*) = \delta + \alpha, \quad h_2^* = 0.$$

For any k_0 near k^*, there exists an optimal solution $k(.)$ that converges to k^* $(t \to +\infty)$.

It is the only solution belonging to BC^1
The result is global.

Remarks

- Turnpike property, policy of investment

Remarks

- Turnpike property, policy of investment
- modified "golden rule"

Complements

Multisectorial case :n-dimensional.

$k(t) \in R^n$

Complements

Multisectorial case :n-dimensional.
$k(t) \in R^n$

- $$\mathcal{L}(t, k(t), \dot{k}(t)) = e^{-\delta t} l(k(t), \dot{k}(t))$$

  with $l : R^n x R^n \rightarrow R$ concave, regular.

Complements

Multisectorial case :n-dimensional.
$k(t) \in R^n$

- $$\mathcal{L}(t, k(t), \dot{k}(t)) = e^{-\delta t} l(k(t), \dot{k}(t))$$

  with $l : R^n x R^n \to R$ concave, regular.
- Euler-Lagrange NS condition of optimality (on BC[1]).

Complements

Multisectorial case :n-dimensional.
$k(t) \in R^n$

- $$\mathcal{L}(t, k(t), \dot{k}(t)) = e^{-\delta t} l(k(t), \dot{k}(t))$$
  with $l : R^n x R^n \to R$ concave, regular.
- Euler-Lagrange NS condition of optimality (on BC[1]).
- An equilibrium $(k^*, 0)$ of the equivalent system to Euler-Lagrange is called *a regular saddle point* if : the linearised system admits as many eigenvalues with real negative part than with real postive part.

Complements

Multisectorial case :n-dimensional.
$k(t) \in R^n$

- $$\mathcal{L}(t, k(t), \dot{k}(t)) = e^{-\delta t} l(k(t), \dot{k}(t))$$
  with $l : R^n x R^n \to R$ concave, regular.
- Euler-Lagrange NS condition of optimality (on BC[1]).
- An equilibrium $(k^*, 0)$ of the equivalent system to Euler-Lagrange is called *a regular saddle point* if : the linearised system admits as many eigenvalues with real negative part than with real postive part.
- Conditions under which a steady state is a regular saddle point ?
  Saddle point stability problem (around 1975 with Pontryagin-Halkin)

Complements ...

With the Pontryagin approach :

- We construct the Hamiltonian $H(k,p)$ and from the Maximum Principle we have

$$\left\{ \begin{array}{l} \dot{k}(t) \\ \dot{p}(t) \end{array} \right\} = \left\{ \begin{array}{l} H_p(k,p) \\ -H_k(k,p) + \delta p \end{array} \right\}$$

Complements ...

With the Pontryagin approach :

- We construct the Hamiltonian $H(k, p)$ and from the Maximum Principle we have

$$\left\{ \begin{array}{l} \dot{k}(t) \\ \dot{p}(t) \end{array} \right\} = \left\{ \begin{array}{l} H_p(k, p) \\ -H_k(k, p) + \delta p \end{array} \right\}$$

- Equivalent to the Euler-lagrange equation.

Complements ...

With the Pontryagin approach :

- We construct the Hamiltonian $H(k,p)$ and from the Maximum Principle we have

$$\left\{ \begin{array}{l} \dot{k}(t) \\ \dot{p}(t) \end{array} \right\} = \left\{ \begin{array}{l} H_p(k,p) \\ -H_k(k,p) + \delta p \end{array} \right\}$$

- Equivalent to the Euler-lagrange equation.
- (k^*, p^*) a steady state is called a *regular saddle point* if
  - n eigenvalues $\lambda_i, i = 1, 2, ...n$ with $Re(\lambda_i) < 0$
  - n eigenvalues $\lambda_i, i = n+1, ..., 2n$ with $Re(\lambda_i) < 0$

Complements ...

With the Pontryagin approach :

- We construct the Hamiltonian $H(k,p)$ and from the Maximum Principle we have

$$\left\{ \begin{array}{l} \dot{k}(t) \\ \dot{p}(t) \end{array} \right\} = \left\{ \begin{array}{l} H_p(k,p) \\ -H_k(k,p) + \delta p \end{array} \right\}$$

- Equivalent to the Euler-lagrange equation.
- (k^*, p^*) a steady state is called a *regular saddle point* if
  - n eigenvalues $\lambda_i, i = 1, 2, ...n$ with $Re(\lambda_i) < 0$
  - n eigenvalues $\lambda_i, i = n+1, ..., 2n$ with $Re(\lambda_i) < 0$
- For the Saddle Point Stability we have to work with the second derivative of the hamiltonian : $H_{kk}, H_{kp}, H_{pk}, H_{pp}$.

Complements ...

With the Pontryagin approach :

- We construct the Hamiltonian $H(k,p)$ and from the Maximum Principle we have

$$\left\{ \begin{array}{l} \dot{k}(t) \\ \dot{p}(t) \end{array} \right\} = \left\{ \begin{array}{l} H_p(k,p) \\ -H_k(k,p) + \delta p \end{array} \right\}$$

- Equivalent to the Euler-lagrange equation.
- (k^*, p^*) a steady state is called a *regular saddle point* if
  - n eigenvalues $\lambda_i, i = 1, 2, ...n$ with $Re(\lambda_i) < 0$
  - n eigenvalues $\lambda_i, i = n+1, ..., 2n$ with $Re(\lambda_i) < 0$
- For the Saddle Point Stability we have to work with the second derivative of the hamiltonian : $H_{kk}, H_{kp}, H_{pk}, H_{pp}$.
- $(k,p) \rightarrow H(k,p)$ concave in k and convex in p.
  ... lake of informations on the preceding matrix.

Lagrangian

With our Lagrangian approach :

- $\mathcal{L}_{kk}, \mathcal{L}_{k\dot{k}}, \mathcal{L}_{\dot{k}k}, \mathcal{L}_{\dot{k}\dot{K}}$

Lagrangian

With our Lagrangian approach :

- $\mathcal{L}_{kk}, \mathcal{L}_{k\dot{k}}, \mathcal{L}_{\dot{k}k}, \mathcal{L}_{\dot{k}\dot{K}}$
- $(k, \dot{k}) \rightarrow \mathcal{L}(k, \dot{k})$ is concave in all variables.
  Particularly we obtain relations with the cross derivatives

Lagrangian

With our Lagrangian approach :

- $\mathcal{L}_{kk}, \mathcal{L}_{k\dot{k}}, \mathcal{L}_{\dot{k}k}, \mathcal{L}_{\dot{k}\dot{K}}$
- $(k, \dot{k}) \rightarrow \mathcal{L}(k, \dot{k})$ is concave in all variables.
  Particularly we obtain relations with the cross derivatives
- For instance, we can prove :
  If $[\mathcal{L}_{k\dot{k}} + \mathcal{L}_{\dot{k}k}](k^*, 0)$ is semi definite positive then we have a regular saddle point.

Lagrangian

With our Lagrangian approach :

- $\mathcal{L}_{kk}, \mathcal{L}_{k\dot{k}}, \mathcal{L}_{\dot{k}k}, \mathcal{L}_{\dot{k}\dot{K}}$
- $(k, \dot{k}) \to \mathcal{L}(k, \dot{k})$ is concave in all variables.
  Particularly we obtain relations with the cross derivatives
- For instance, we can prove :
  If $[\mathcal{L}_{k\dot{k}} + \mathcal{L}_{\dot{k}k}](k^*, 0)$ is semi definite positive then we have a regular saddle point.
- Existence of cycles ....

Lagrangian

With our Lagrangian approach :

- $\mathcal{L}_{kk}, \mathcal{L}_{k\dot{k}}, \mathcal{L}_{\dot{k}k}, \mathcal{L}_{\dot{k}\dot{K}}$
- $(k, \dot{k}) \rightarrow \mathcal{L}(k, \dot{k})$ is concave in all variables.
  Particularly we obtain relations with the cross derivatives
- For instance, we can prove :
  If $[\mathcal{L}_{k\dot{k}} + \mathcal{L}_{\dot{k}k}](k^*, 0)$ is semi definite positive then we have a regular saddle point.
- Existence of cycles ....
- With some other space of curves :
  ... Euler-Lagrange + transversality condition.

Hartman-Grobman theorem

$$\dot{x} = f(x(t))$$

$f : \Re^n \to \Re^n \quad C^1$
$\bar{x} \quad f(\bar{x}) = 0$

$$\dot{X} = df(\bar{x})X$$

$\bar{x}$ hyperbolic : $\forall \lambda,\ Re(\lambda) \neq 0$

In a neighborhood of $\bar{x}$, the two dynamics are homeomorphic with conservation of the orientation of the trajectories.

Nemytski operator

$$f : \subset \Re x \Re^n \longrightarrow \Re^m$$

$$\mathcal{N}_f : \ C^0([t_0, t_1], \Re^n) \longrightarrow C^0([t_0, t_1], \Re^m$$

$$\mathcal{N}_f(x(.)) = f(., x(.))$$

If $f(.)$ is C^1 then $\mathcal{N}_f$ too and

$$\mathcal{N}_f'(x(.)).h(.) = f_x'(., x(.)).h(.)$$