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A Stochastic Framework for Socio-Economic Interactions

Consider a population with an binary set of choices or opinions (denoted by "$+$" and "$-$"):

$$n_+ + n_- = 2N \tag{1}$$

The *socio-economic configuration* at any point in time can be characterized by:

$$n = \frac{1}{2}(n_+ - n_-) \tag{2}$$

or the *opinion index* x:

$$x = \frac{n}{N}, \qquad x \in [-1, 1] \tag{3}$$

Note that $n_+ = N + n$, $n_- = N - n$.

We assume that the dynamics can be captured by certain *transition probabilities* for agents to move from the "$+$" to the "$-$"group and vice versa, p_{-+} and p_{+-}. This means that the population composition follows a stochastic process (which might have systematic components entering the specifications of p_{-+} and p_{+-}). Note that this framework is different from the more static probabilistic choice in the Discrete Choice model. A complete characterization of the process requires to solve for the probability distribution at any point in time, t, over all possible states, n or x:

$$P(n;t) \qquad \text{with} \qquad \sum_{n=-N}^{N} P(n;t) = 1 \tag{4}$$

or

$$P(x;t) \qquad \text{with} \qquad \sum_{x=-1}^{1} P(x;t) = 1 \tag{5}$$

These time-dependent distributions might or might not converge to a *stationary distribution* for $t \to \infty$.

Dynamics: the probabilities to find the system in states n (or x) change over time according to the probabilities for movements of single individuals. For example, the configuration might change from n to $n+1$ or $n-1$ with one agent moving to another group with probabilities:

$$w(n \to n+1) = n_- p_{+-}(n;t) = (N-n)p_{+-}(n), \tag{6}$$

$$w(n \to n-1) = n_+ p_{-+}(n;t) = (N+n)p_{-+}(n) \tag{7}$$

assuming that there is an influence of the overall configuration n on individuals' propensities to move between groups and that it is only the overall number of agents in the " + " and " − " groups, that influences these decisions.

Our goal is to obtain insights into the *macroscopic* behavior of a system of many interacting agents (in terms of the time change of n or x) from its microscopic properties, i.e. from the hypotheses on the systematic components of the movements between groups of individual agents.

We assume that time is continuous and that individual switches can be formalized via Poisson processes. We can, then, specify the dynamic process more formally via conditional probabilities,e.g. $\omega(n+1, t+\Delta t \mid n, t)$, $\omega(n-1, t+\Delta t \mid n, t)$, $\omega(n+2, t+\Delta t \mid n, t)$ etc. and in the limit $\triangle t \to 0$ we define [1]

[1] The probability to observe n realizations of a Poisson process within a time interval Δt is given by

$$P_n(\Delta t) = \frac{(\lambda \Delta t)^n}{n!} e^{-\lambda \Delta t}. \tag{8}$$

Hence,

$$\lim_{\Delta t \to 0} \frac{\omega(n+1, t+\Delta t \mid n, t)}{\Delta t} = w(n+1 \mid n, t) \tag{11}$$

and

$$\lim_{\Delta t \to 0} \frac{\omega(n-1, t+\Delta t \mid n, t)}{\Delta t} = w(n-1 \mid n, t) \tag{12}$$

etc. It follows from the Poisson probabilities (9) and (10) that for $\Delta t \to 0$ more than two simultaneous movements of individuals become increasingly unlikely and the probability for one individual to change his mind converges to $\lambda \Delta t$ with λ the *transition rate* of the Poisson process (8) which in our framework is given by the expressions introduced in eqs. (6) and (7): $w(n+1 \mid n, t) = w(n \to n+1) \equiv w_{\uparrow}(n) = n_- p_{+-}$ and $w(n-1 \mid n, t) = w(n \to n-1) \equiv w_{\downarrow}(n) = n_+ p_{-+}$.

The Master Equation for the Time Change of the Probability Density

The overall evolution of the system has to be described by the time change of the probabilities over all states. In general, this amounts to a system of difference equations for all possible system configurations n:

$$P(n, t+\Delta t) - P(n, t) = \tag{13}$$

$$\underbrace{\sum_{n'} \omega(n, t+\Delta t \mid n', t) P(n'; t)}_{\text{inflow of probability to state n}} - \underbrace{\sum_{n'} \omega(n', t+\Delta t \mid n, t) P(n; t)}_{\text{outflow of probability from state n}}$$

which is called the *Master equation* for the probability flux. In the continuous-time limit we obtain:

$$\frac{dP(n;t)}{dt} = \sum_{n'} \{ w(n \mid n', t) P(n'; t) - w(n' \mid n, t) P(n, t) \} \tag{14}$$

$$P_1(\Delta t) = \lambda \Delta t e^{-\lambda \Delta t} = \lambda \Delta t (1 - \lambda \Delta t + \frac{(\lambda \Delta t)^2}{2} - \ldots) \tag{9}$$

$$P_2(\Delta t) = \frac{(\lambda \Delta t)^2}{2} e^{-\lambda \Delta t} \quad etc. \tag{10}$$

Note that in the continuous-time limit, transition probabilities $\omega(\cdot)$ have been replaced by transition rates $w(\cdot)$ on the right-hand side of eq.(14)

While the Master equation is more generally applicable, in our case of Poisson jump processes we can restrict attention to neighboring states $n' = n \pm 1$ so that:

$$\begin{aligned} \frac{dP(n;t)}{dt} &= w_{\downarrow}(n+1)P(n+1;t) + w_{\uparrow}(n-1)P(n-1;t) \\ &\quad -(w_{\uparrow}(n) + w_{\downarrow}(n))P(n;t) \end{aligned} \tag{15}$$

The Master equation can be solved analytically only in exceptional cases, but it can always be simulated.

Assuming that the transition probabilities of individuals do not depend on the raw numbers, but rather on the fraction of members of both groups, we can also express the dynamics in terms of the opinion index x. Since $\Delta n = 1$ corresponds to $\Delta x = \frac{\Delta n}{N} = \frac{1}{N}$, we have:

$$\begin{aligned} w_{\uparrow}(x) = w(x + \frac{1}{N} \mid x, t) &= n_{-}p_{+-}(x) \\ &= N(1-x)p_{+-}(x) \end{aligned} \tag{16}$$

$$\begin{aligned} w_{\downarrow}(x) = w(x - \frac{1}{N} \mid x, t) &= n_{+}p_{-+}(x) \\ &= N(1+x)p_{-+}(x) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \frac{dP(x;t)}{dt} &= w_{\downarrow}(x + \frac{1}{N})P(x + \frac{1}{N};t) + w_{\uparrow}(x - \frac{1}{N}) \\ &\quad P(x - \frac{1}{N};t) - (w_{\uparrow}(x) + w_{\downarrow}(x))P(x;t) \end{aligned} \tag{18}$$

which gives the Master equation for the intensive macrovariable x in continuous time.

The Fokker-Planck Equation

Since for large N, x is close to a continuous quantity, we can perform a Taylor series expansion with respect to Δx: Rearranging (18) gives

$$\begin{aligned} \frac{dP(x;t)}{dt} &= w_{\uparrow}(x-\frac{1}{N})P(x-\frac{1}{N};t) - w_{\uparrow}(x)P(x;t) \\ &\quad + w_{\downarrow}(x+\frac{1}{N})P(x+\frac{1}{N};t) - w_{\downarrow}(x)P(x;t) \end{aligned} \tag{19}$$

A second-order approximation of the first and second group of components on the right-hand side around x yields:

$$\begin{aligned} \frac{\partial P(x;t)}{\partial t} &= \frac{\partial}{\partial x}[w_{\uparrow}(x)P(x;t)](-\frac{1}{N}) \\ &\quad + \frac{1}{2}\frac{\partial}{\partial x^2}[w_{\uparrow}(x)P(x;t)](-\frac{1}{N})^2 \\ &\quad + \frac{\partial}{\partial x}[w_{\downarrow}(x)P(x;t)]\frac{1}{N} + \frac{1}{2}\frac{\partial}{\partial x^2}[w_{\downarrow}(x)P(x,t)](\frac{1}{N})^2 \end{aligned} \tag{20}$$

$$\begin{aligned} \Rightarrow \frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}[(w_{\uparrow}(x) - w_{\downarrow}(x))P(x;t)]\frac{1}{N} \\ &\quad + \frac{1}{2}\frac{\partial}{\partial x^2}[(w_{\uparrow}(x) + w_{\downarrow}(x))P(x;t)]\frac{1}{N^2} \end{aligned} \tag{21}$$

This is the so-called *Fokker-Planck Equation* which contains information on the

- drift (systematic motion): $A(x) = \frac{1}{N}(w_{\uparrow}(x) - w_{\downarrow}(x))$
- diffusion (fluctuations): $D(x) = \frac{1}{N^2}(w_{\uparrow}(x) + w_{\downarrow}(x))$

Note that in the above framework a law of large numbers applies since $D(x) \rightarrow 0$ for $N \rightarrow \infty$.[2]

The Fokker-Planck equation can often be solved analytically and leads to a Gaussian approximation of the probability distribution at time t (since it only covers first and second moments). Note, however, that this Gaussian shape is not maintained during the temporal development but might change into a variety of other shapes depending on the precise laws of motion (cf. our examples

[2] This depends on the formalisation of transition probabilities for individuals.

below). The advantages of the Fokker-Planck equation over the exact Master equation are that it can be simulated more easily , and that in certain cases (for simple drift and diffusion processes) it is possible to derive the stationary distribution from the Fokker-Planck equation by setting the left-hand side $\frac{\partial P(x;t)}{\partial t} = 0$. If the distribution remains constant over time, we obviously obtain a characterization of the limit state the system converges to. Solving for the stationary distribution *P(x)* from the right-hand side of equation (21) would only require integration with respect to x which might be possible in cases with simple formalizations of transition rates.

Macroscopic Dynamics

One obtains further insights into the dynamics by derivation of *macroscopic quantities*, e.g. the mean, variance etc.

For the mean $\overline{x}_t$ note that its time change can be computed using the Master equation:

$$\overline{x}_t = \sum_{x=-1}^{1} xP(x;t) \tag{22}$$

$$\frac{d\overline{x}_t}{dt} = \sum_{x=-1}^{1} x\frac{dP(x;t)}{dt} \tag{23}$$

In the general case with jumps of arbitrary size, we get:

$$\frac{d\overline{x}_t}{dt} = \sum_{x} x \sum_{x'} (w_{xx'}P(x',t) - w_{x'x}P(x;t)) \tag{24}$$

with $w_{xx'}$ denoting the transition rates for movements from x' to x. Since the summation for x and x' is over the same set of values, we can exchange the order of summation:

$$
\begin{aligned}
\frac{d\overline{x_t}}{dt} &= \sum_x \sum_{x'} x w_{xx'} P(x';t) - \sum_x \sum_{x'} x w_{x'x} P(x;t) && (25)\\
&= \sum_x \sum_{x'} x' w_{x'x} P(x;t) - \sum_x \sum_{x'} x w_{x'x} P(x;t) && (26)\\
&= \sum_x \underbrace{\sum_{x'} (x'-x) w_{x'x}}_{\equiv a_{x,1}} P(x,t) && (27)\\
&= \sum_x a_{x,1} P(x;t) = \overline{a_{x,1}} && (28)
\end{aligned}
$$

$a_{x,1}$ is denoted the *first jump moment* and in our case is given by:

$$
\begin{aligned}
\sum_{x'} (x'-x) w_{x'x} &= \frac{1}{N} w_{\uparrow}(x) + (-\frac{1}{N}) w_{\downarrow}(x) && (29)\\
&= \frac{1}{N}(w_{\uparrow}(x) - w_{\downarrow}(x)) && (30)
\end{aligned}
$$

which recovers the drift term from the Fokker-Planck equation.

The change of the mean value is, therefore, determined by the average jump of the system from any realisation x of the opinion index to its neighboring values weighted by its probability of occurence, i.d. the probability distribution of x at time t, $P(x;t)$:

$$
\frac{d\overline{x}_t}{dt} = \overline{a_{x,1}(x)} \tag{31}
$$

If $a_{x,1}$ is linear, we arrive at an exact mean value equation in closed form:

$$
\frac{d\overline{x}_t}{dt} = a_{x,1}(\overline{x}) \tag{32}
$$

If $a_{x,1}$ is non-linear, we can perform an approximation of the first jump moment $a_{x,1}$ around the most probable value, $\overline{x}$. Replacing the bars by the expectation operator provides a more transparent representation of the resulting Taylor series expansion:

$$\frac{d\overline{x}_t}{dt} = E[a_{x,1}(x)] = E[a_{x,1}(\bar{x}) + (x - \bar{x})a'_{x,1} + \frac{1}{2}(x - \bar{x})^2 a''_{x,1}(\bar{x}) + ...] \tag{33}$$

Obviously, the second term vanishes in expectation, so that we arrive at:

$$\frac{d\overline{x}_t}{dt} = a_{x,1}(\overline{x}) + \underbrace{(\overline{x - \overline{x}})}_{=0} a'_{x,1}(\overline{x}) + \frac{1}{2} \underbrace{(\overline{x - \overline{x})^2}}_{\sigma_x^2} a''_{x,1}(\overline{x}) + \dots \tag{34}$$

We can, then, express the differential equation for $\bar{x}_t$ as the sum of two distinct terms with different interpretation:

$$\frac{d\overline{x}_t}{dt} = \underbrace{a_{x,1}(\overline{x})}_{\text{pure mean value dynamics}} + \underbrace{\frac{1}{2}\sigma_x^2 a''_{x,1}(\overline{x})}_{\text{second-order correction for influence of fluctuations}} \tag{35}$$

Examples:

(1) A simple birth-death process: while we have developed the above formal apparatus for a model with two groups, the introduction of the difference n between group occupation numbers reduces the dynamics to that of one macrovariable (n or x). This is formally equivalent to the description of the group size dynamics in the case of one single group with occupation number n with a maximum N of group members. As a first example, we investigate a simple process of the change of the size of a group through birth and death dynamics. Let us assume that the birth rate is constant and equal to λ so that new group members emerge according to a Poisson process as offspring from existing group members:

$$w(n \to n + 1) = \lambda n \equiv r_n. \tag{36}$$

Obviously, this assumes that all existing group members have the same fertility and reproduction is asexual.
For the death rate, in contrast, we assume that it is not constant but increases with the population size due, for example, to exhaustion of available resources. As the simplest formalization we assume a linear relationship with flexible Poisson rates $\mu\frac{n}{N}$. N would, then, be a measure

for the maximum population size or carrying capacity of the environment. From the transition rates from states n in the lower ones this leads to:

$$w(n \to n-1) = \mu \frac{n}{N} n \equiv l_n. \tag{37}$$

Using the short-hand notations r_n and l_n (for *rightward* and *leftward* moves), we can establish the Master equation in continuous time:

$$\frac{dP(n,t)}{dt} = r_{n-1}P(n-1;t) + l_{n+1}P(n+1;t) - (r_n + l_n)P(n;t). \tag{38}$$

The first jump moment for n is:

$$a_{n,1} = \sum_{n'} (n' - n) w(n \to n') = 1 \cdot r_n + (-1) \cdot l_n = \lambda n - \frac{\mu}{N} n^2, \tag{39}$$

so that we arrive at the mean-value dynamics:

$$\frac{d\overline{n_t}}{dt} = \overline{a_{n,1}(n)} = \overline{\lambda_n - \frac{\mu}{N} n^2}. \tag{40}$$

In first-order approximation, we get a simple differential equation in $\overline{n_t}$:

$$\frac{d\overline{n_t}}{dt} \simeq \lambda \overline{n_t} - \frac{\mu}{N} \overline{n_t^2} \tag{41}$$

Of course, we could also define an intensive variable analogous to our previous index $x : x = \frac{n}{N}$ which now gives the relative size of the current population compared to the carrying capacity N. We note that the transition rates for movements of x to its neighboring states $x \pm \frac{1}{N}$ are the same as in (36) and (37) so that $r_n = r_x$ and $l_n = l_x$ hold. The Master equation for the intensive variable is obtained as:

$$\frac{dP(x,t)}{dt} = r_{x-\frac{1}{N}} P(x - \frac{1}{N}; t) + l_{x+\frac{1}{N}} P(x + \frac{1}{N}; t) - (r_x + l_x) P(x; t) \tag{42}$$

The first jump moment for x is:

$$a_{x,1} = \sum_{x'}(x' - x)w(x \to x') = \frac{1}{N}\lambda n + (-\frac{1}{N})\frac{\mu}{N}n^2 = \lambda x - \mu x^2. \quad (43)$$

Deriving the mean value dynamics for x, we obtain as a first-order approximation:

$$\frac{d\overline{x_t}}{dt} = \lambda\overline{x_t} - \mu\overline{x_t}^2 \quad (44)$$

which is known as the logistic function. It is easy to get some basic insights from (44). *Equilibria* require $\frac{d\overline{x_t}}{dt} = 0$ which leads to $\lambda\overline{x} = \mu\overline{x}^2$. We, therefore, find two possible equilibria $\overline{x_0}^* = 0$ and $\overline{x_1}^* = \frac{\lambda}{\mu}$. One can easily check that they correspond to equilibria of the n-dynamics $\overline{n_0}^* = 0$ and $\overline{n_1}^* = \frac{\lambda N}{\mu}$.
Equilibria of a first-order differential equation are (locally) stable if the first derivative of the dynamic law $\frac{d\overline{x_t}}{dt} = f(\overline{x})$ is negative at the equilibrium. Since $f'(\overline{x}) = \lambda - 2\mu\overline{x}$, we see that $\overline{x_0}^*$ is unambiguously unstable while $\overline{x_1}^*$ is unambiguously stable: the system will tend towards a long-run population ratio of $\frac{\lambda}{\mu}$ of the maximum capacity N. Fig. 1 shows the first-order approximation of the mean-value dynamics together with a simulation of the microscopic dynamics of the birth-death process. As it can be seen, the group size quickly converges to the neighborhood of the steady state of the mean value and after this short transient period exhibits only minor fluctuations in the neighborhood of $\overline{x_1}^*$. Fig. 2 shows the pertinent development of the entire probability distribution (simulated via the Fokker-Planck equation) which shows the shift of probability mass over time from the initial condition towards $\overline{x_1}^*$.

It is worthwhile to remember that despite the apparently good prediction of the average behavior from the stable root of the approximate mean-value dynamics (44), the resulting $\overline{x_1}^*$ is *not* the exact mean value of the process. With higher-order terms included in the approximation, one would get successively closer to the true value. The next higher approximation would allow for the second-order correction of eq. (40) which, for the dynamics of $\overline{x_t}$, would lead to:

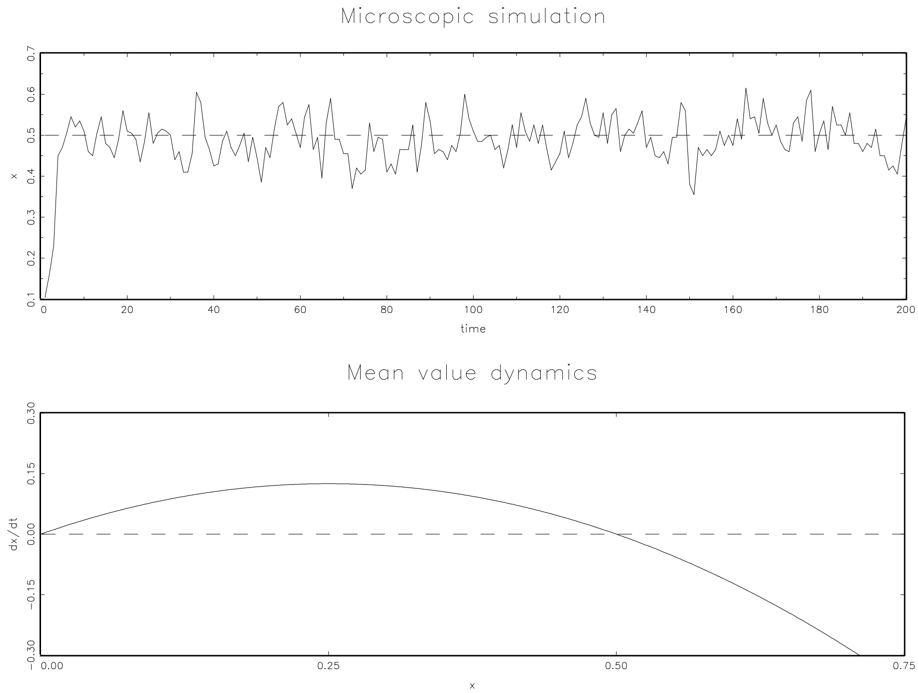

Figure 1: Mean value dynamics and simulated microscopic process. Parameters are $\lambda = 1, \mu = 2, N = 200$.

$$\frac{d\overline{x_t}}{dt} = \lambda\overline{x_t} - \mu\overline{x_t}^2 - \mu\sigma_x^2 \tag{45}$$

since $a''_{x,2}(x) = -2\mu$. In order to implement (45), we would, however, need information on the dynamics of the fluctuations around the mean, σ_x^2.

(2) As our second example, we consider a pure social dynamics in which individuals have a higher propensity to switch to the larger group. A popular formalization is:

$$p_{+-} = \nu e^{\alpha x}, p_{-+} = \nu e^{-\alpha x} \tag{46}$$

with: ν: general frequency of jumps,
α: intensity of social influence.

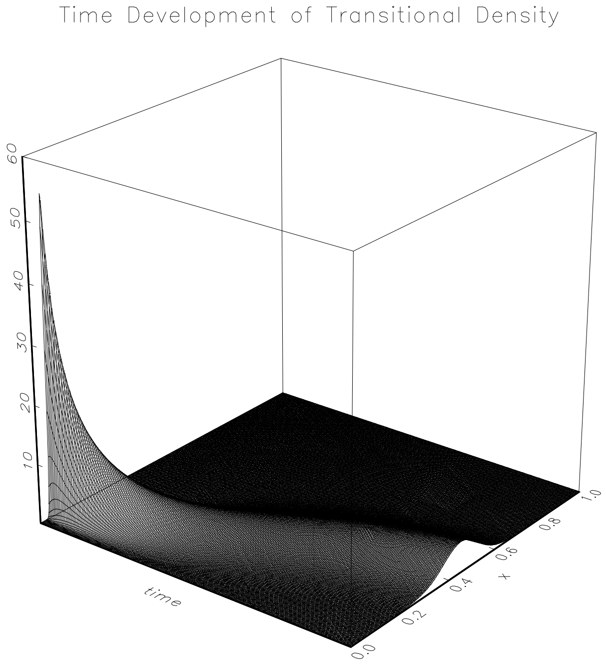

Figure 2: The time change of the probability distribution from a numerical solution of its Fokker-Planck equation, same parameters as in Fig. 1 with initial condition of $n = 10$.

It follows that:

$$w_\uparrow(x) = N(1-x)\nu e^{\alpha x}, \quad w_\downarrow(x) = N(1+x)\nu e^{-\alpha x} \tag{47}$$

$$\begin{aligned} a_{x,1} &= \sum_{x'} (x'-x) w_{x'x} \\ &= \frac{1}{N}[N(1-x)\nu e^{\alpha x} - N(1+x)\nu e^{-\alpha x}] \qquad (48) \\ &= (1-x)\nu e^{\alpha x} - (1+x)\nu e^{-\alpha x} \\ &= 2\nu(\sinh(\alpha x) - x\cosh(\alpha x)) \\ &= 2\nu(\tanh(\alpha x) - x)\cosh(\alpha x) \qquad (49) \end{aligned}$$

where we made use of the definitions of the hyperbolic trigonometric functions:

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}), \quad (50)$$

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}), \quad (51)$$

$$\tanh(x) = \frac{\sinh(x)}{cosh(x)}. \quad (52)$$

The first-order approximation to the mean-value dynamics yields:

$$\frac{d\overline{x}_t}{dt} = 2\nu(\tanh(\alpha\overline{x}) - \overline{x})\cosh(\alpha\overline{x}). \quad (53)$$

Since $\cosh(x) > 0$ holds for all x, the condition for an equilibrium of the mean value dynamics becomes:

$$\frac{d\overline{x}_t}{dt} = 0 \Rightarrow \overline{x}^* = \tanh(\alpha\overline{x}^*) \quad (54)$$

since $\tanh(x)$ is bounded between -1 and 1 and its local slope at 0 is 1, we get the following structure of equilibria:

- $\alpha \leq 1$ implies existence of a unique equilibrium $\overline{x}^* = 0$,
- $\alpha > 1$ leads to multiple equilibria: $\overline{x}^*_-$, $\overline{x}^*_0$, $\overline{x}^*_+$ with $\overline{x}^*_+ = -\overline{x}^*_-$

As a generalization of this framework, we could allow for a general positive or negative bias α_0 towards one alternative:

$$p_{+-} = \nu e^{\alpha_0 + \alpha_1 x}, \quad p_{-+} = \nu e^{-\alpha_0 - \alpha_1 x} \quad (55)$$

with $\alpha_0 \neq 0$ one would still get a unique (multiple) equilibria for $\alpha_1 \leq 1 (\alpha_1 > 1)$. The resulting equilibrium distributions would be symmetrically bi-modal in the case $\alpha_0 = 0$, $\alpha_1 > 1$ with equal probabilities of a "+" or "-" majority. In the case of a biased herding process, $\alpha_1 > 1$ still implies bi-modality in most cases, albeit with an asymmetry both for the location of $\overline{x_+}^*$ and $\overline{x_-}^*$ and their probabilities of occurrence. If α_0 becomes large in absolute value, the bias eventually becomes the dominant factor and interaction effects play only a minor role, so that the system behavior switches back from bi-modality to uni-modality (i.e., a unique equilibrium of the mean value dynamics). Fig. 3 illustrates the determination of equilibria in the intersection of the *tanh* function with the 45° line, while Fig. 4 shows the development of the transient density

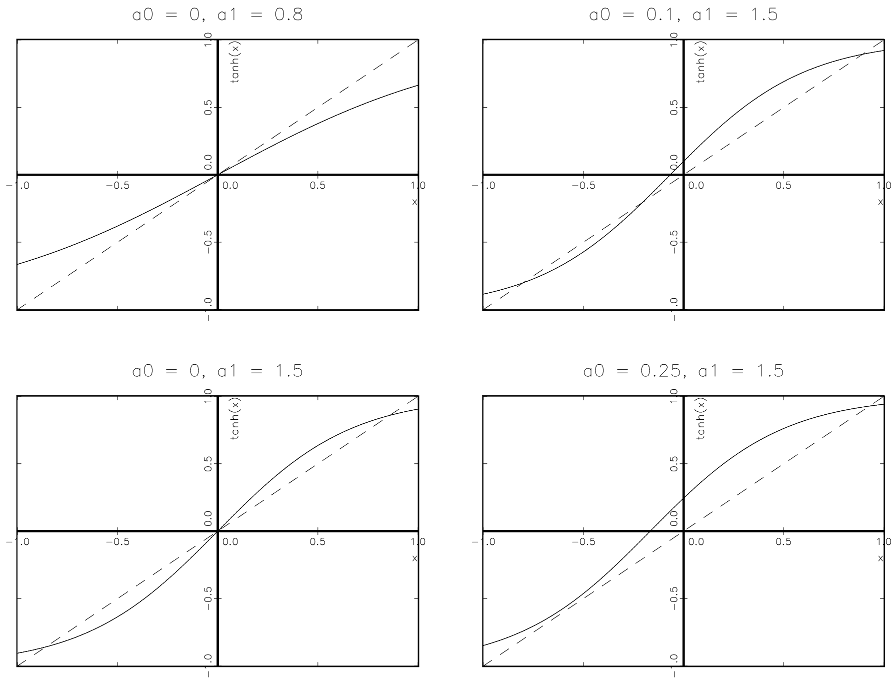

Figure 3: Location of equilibria of the social opinion dynamics

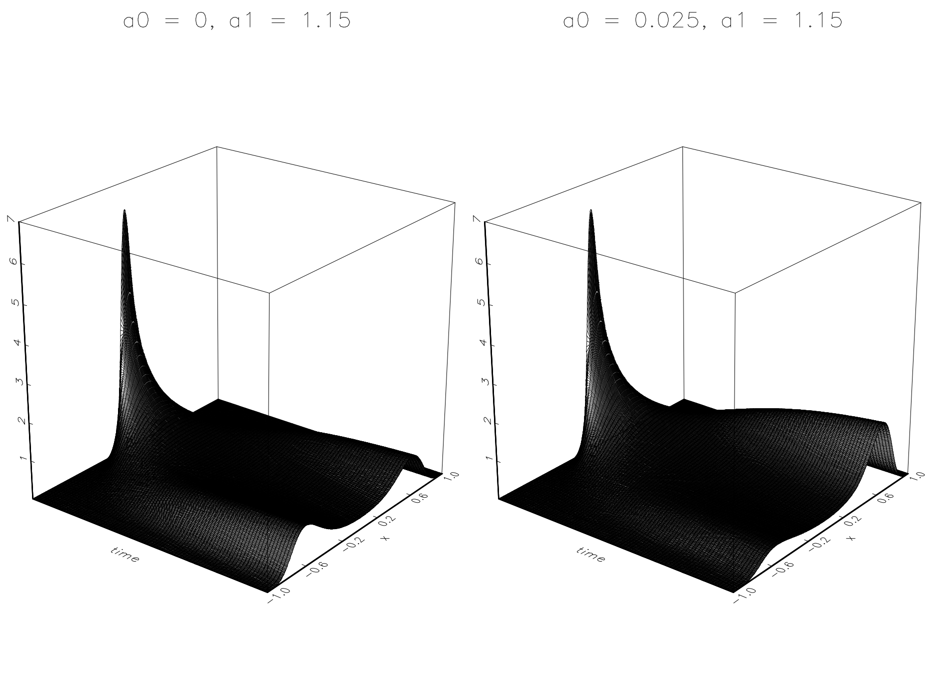

Figure 4: Transient densities of the social opinion dynamics

for selected cases.

For the second-order approximation derived in eq. (35), we get:

$$\frac{d\overline{x}_t}{dt} = 2\nu(\sinh(\alpha\overline{x}) - \overline{x})\cosh(\alpha\overline{x}) + \nu((\alpha^2 - 2\alpha)\sinh(\alpha\overline{x}) - \overline{x}\alpha\cosh(\alpha\overline{x}))\sigma_x^2 \tag{56}$$

since

$$a''_{x,1}(\overline{x}) = 2\nu((\alpha^2 - 2\alpha)\sinh(\alpha\overline{x}) - \alpha\overline{x}\cosh(\alpha\overline{x})). \tag{57}$$

Dynamics of Higher Moments

We now move on to determine the dynamics of the second moment:

$$\overline{x_t^2} = \sum_x x^2 P(x;t) \tag{58}$$

Its change in time can be investigated along the same lines as with mean value dynamics using the Master equation in continuous time:

$$\frac{d}{dt}\overline{x_t^2} = \sum_x x^2 \frac{dP(x;t)}{dt} \tag{59}$$

$$= \sum_x x^2 \sum_{x'} (w_{xx'}P(x';t) - w_{xx'}P(x;t)) \tag{60}$$

$$= \sum_x \sum_{x'} (x'^2 - x^2) w_{x'x} P(x;t) \tag{61}$$

$$= \sum_x \sum_{x'} ((x' - x)^2 + 2x(x' - x)) w_{x'x} P(x;t) \tag{62}$$

so that with

$$\sum_{x'} (x' - x)^2 w_{x'x} = a_{x,2} \quad : \text{the second jump moment}, \tag{63}$$

we arrive at:

$$\frac{d\overline{x_t^2}}{dt} = \sum_x (a_{x,2} + 2x a_{x,1}) P(x;t) \tag{64}$$

$$= \overline{a_{x,2}} + 2\overline{x a_{x,1}} \tag{65}$$

The variance is defined as $\sigma_x^2 = \overline{x^2} - \overline{x}^2$ so that its change in time is

$$\begin{aligned} \frac{d}{dt}\sigma_x^2 &= \frac{d}{dt}(\overline{x^2} - \overline{x}^2) && (66)\\ &= \frac{d}{dt}\overline{x^2} - \frac{d}{dt}\overline{x}^2 && (67)\\ &= \overline{a_{x,2}} + 2\overline{x a_{x,1}} - 2\overline{x}\ \overline{a_{x,1}} && (68)\\ &= \overline{a}_{x,2} + 2\overline{(x-\overline{x})a_{x,1}} && (69) \end{aligned}$$

Performing a Taylor series expansion around $\overline{x}$ for both terms on the right-hand side of (69) gives:

$$\begin{aligned} \frac{d}{dt}\sigma_x^2 &= E[a_{x,2}(\overline{x}) + (x-x)a'_{x,2}(\overline{x}) + ... && (70)\\ &+2(x-\overline{x})a_{x,1}(\overline{x}) + 2(x-\overline{x})(x-\overline{x})a'_{x,1} + ...] && (71) \end{aligned}$$

where we have replaced the outer bars by the expectation operator for better readability. Obviously, the second and third term in this expansion are zero in expectation, so that to first-order accuracy we only need to keep track of the two remaining terms:

$$\begin{aligned} \frac{d}{dt}\sigma_x^2 &\approx a_{x,2}(\overline{x}) + 2\overline{(x-\overline{x})^2}a'_{x,1}(\overline{x}) + \ldots && (72)\\ &\simeq a_{x,2}(\overline{x}) + 2\sigma_x^2 a'_{x,1}(\overline{x}). && (73) \end{aligned}$$

If both $a_{x,1}$ and $a_{x,2}$ are linear, the last line is again exact, if not, it is an approximation up to the first order.

Examples:

(1) Birth-death process

The second jump moment of the above birth-death process is:

$$\begin{aligned} a_{x,2} &= \sum_{x'}(x'-x)^2 w(x \to x') = && (74)\\ &= \frac{1}{N^2}\lambda n + (-\frac{1}{N})^2\mu\frac{n}{N}n = \frac{1}{N}(\lambda x + \mu x^2) && (75) \end{aligned}$$

We, therefore, arrive at the first-order approximation of the variance dynamics:

$$\frac{d}{dt}\sigma_x^2 = \frac{1}{N}(\lambda\overline{x} + \mu\overline{x}^2) + 2\sigma_x^2(\lambda - 2\mu\overline{x}) \tag{76}$$

This equation provides us with a measure of the dispersion of single realizations of the microscopic dynamics around its expectation which can be derived from the mean value dynamics. If we set $\frac{d}{dt}\sigma_x^2 = 0$, we obtain the variance of the stationary distribution. Since we already know that the mean value of the stationary distribution is given by $x^* = \frac{\lambda}{\mu}$ in the first-order approximation, we can use this information to substitute for $\overline{x}$ in equilibrium:

$$\frac{d}{dt}\sigma_x^2 = 0 \Rightarrow \frac{1}{N}(\lambda\overline{x} + \mu\overline{x}^2) = -2\sigma_x^2(\lambda - 2\mu\overline{x}) \tag{77}$$

$$\Rightarrow \sigma_x^2 = \frac{-(\lambda\overline{x} + \mu\overline{x}^2)}{2N(\lambda - 2\mu\overline{x})} \tag{78}$$

Inserting the stationary solution $\overline{x}^* = \frac{\lambda}{\mu}$, we obtain the stationary variance $\sigma_x^2 = \frac{\lambda}{N\mu}$. As expected, fluctuations decrease with system size N. This relationship also shows that except for very small N, the second-order correction to the mean value would be relatively small. As an example, with the numbers of our illustrations in Fig. 1 and 2, $\lambda = 1$, $\mu = 2$ and $N = 200$, simultaneous solution of the second-order approximation to the mean-value dynamics and the first-order approximation to the variance dynamics in the stationary state (i.e. for $\frac{d}{dt}x_t = \frac{d}{dt}\sigma_x^2 = 0$) yields $\overline{x}^* \approx 0.4949$ while the value obtained for the stationary variance changes from $\sigma_x^2 = 0.0025$ to $\sigma_x^2 \approx 0.002513$ when using this improved approximation to the mean rather than the first-order approximation $\bar{x}^* = 0.5$. For $N = 20$, the pertinent numbers are $\bar{x}^* \approx 0.4375$ to second-order accuracy (against 0.5 to first order) for the mean and 0.02734 (against 0.025) for the variance.

(2) The herding model

In our second example:

$$a_{x,2} = \sum_{x'}(x' - x)^2 w_{x'x} \tag{79}$$

$$= \frac{1}{N^2}(w_{\uparrow}(x) + w_{\downarrow}(x)) \tag{80}$$

which is the diffusion term $D(x)$ from the Fokker-Planck equation. Inserting the individual transition probabilities we obtain:

$$a_{x,2} = \frac{1}{N^2}(N(1-x)e^{\alpha x} + N(1+x)e^{-\alpha x}) \quad (81)$$
$$= \frac{2\nu}{N}(\cosh(\alpha x) - x\sinh(\alpha x)) \quad (82)$$
$$(83)$$

and

$$a'_{x,1} = 2\nu((\alpha - 1)\cosh(\alpha x) - \alpha x \sinh(\alpha x)). \quad (84)$$

We arrive at the following variance dynamics:

$$\frac{d}{dt}\sigma_x^2 = \frac{2\nu}{N}(\cosh(\alpha\overline{x}) - \overline{x}\sinh(\alpha\overline{x})) + 4\nu((\alpha-1)\cosh(\alpha\overline{x}) - \overline{x}\alpha\sinh(\alpha\overline{x}))\sigma_x^2 \quad (85)$$

One derives the variance in a steady state (for $\frac{d}{dt}\sigma_x^2 = 0$)

$$\overline{\sigma_x^2} = \frac{-(\cosh(\alpha\overline{x}) - \overline{x}\sinh(\alpha\overline{x}))}{2N((\alpha-1)\cosh(\alpha\overline{x}) - \alpha\overline{x}\sinh(\alpha\overline{x}))} \quad (86)$$

If $\overline{x} = 0$, this boils down to:

$$\overline{\sigma_x^2}(\overline{x} = 0) = \frac{1}{2N(1-\alpha)} \quad (87)$$

More generally, substituting $\overline{x^*} = \tanh(\alpha\overline{x^*})$, one derives for arbitrary equilibria $\overline{x^*}$:

$$\overline{\sigma_x^2}(\overline{x^*}) = \frac{1}{2N(\cosh^2(\alpha\overline{x^*}) - \alpha)} \quad (88)$$

These relationships provide a number of interesting insights: First, the variance of the stationary distribution is independent of the frequency of individual transitions, ν. The reason is, that ν only governs the speed of the dynamic process, not its qualitative outcome. One could, in fact, simply drop ν by *rescaling time* through $\tau = \nu \cdot t$. Second, computing the derivative with respect to α shows that σ_x^2 increases with α in the unique equilibrium case $\overline{x}^* = 0$ and diverges for $\alpha \to 1$. In the case of multiple equilibria $\overline{x}_+^*, \overline{x}_-^*$ the variance rather decreases with α and diverges at the lower bound. Fig. 5 summarizes the behavior of the mean values and variances of the stationary distribution under variation of α. The increase of the variance with higher α in the uni-modal case is easily explained via an increase in the herding tendency which leads to more and more deviations from the balanced situation. Once stable majorities have emerged, the same phenomenon leads to increased stability of these majorities and, therefore, reduces fluctuations around $\overline{x}_+^*$ and $\overline{x}_-^*$. Note that the stationary variance in this case is *not* the variance of the entire stationary distribution, but due to its derivation as a first-order approximation of the fluctuations around the mean value rather is a measure of the local fluctuations around one of the modes $\overline{x}_+^*$ or $\overline{x}_-^*$ (i.e., it takes *not* into account potential transitions between both equilibria).

The use of variance equations is, however, not restricted to the information they provide on the fluctuations around stationary mean values. The pertinent differential equations can also be used to study the extent of fluctuations during transient dynamics. Fig. 6 shows an example: the five thin solid lines are replications of microscopic simulations of the herding model with parameters $\nu = 1, \alpha = 1.2$´and $N = 500$. In all cases, the starting value is $x_0 = -0.04$, i.e. a very small dominance of "-" over "+" agents. From our theoretical analysis we know that with the strong herding intensity we are in the regime of bi-modal distributions, so that the system would tend towards $\overline{x}_+^*$ or $\overline{x}_-^*$. The slight dominance of the "-" group gives the second equilibrium a higher chance of occurrence. This is reflected in the mean value dynamics (the thick solid line) which predicts a convergence towards $\overline{x}_-^* \approx -0.66$. However, because of the proximity of the initial condition to 0, this prediction comes with a high uncertainty as small fluctuations could easily drive the microscopic dynamics into the attracting area of $\overline{x}_+^*$ instead. Our five simulations underscore this aspect: two of them converge to $\overline{x}_+^*$ and three to the expected long-run outcome $\overline{x}_-^*$. However, this uncertainty is captured in the temporal development of the variance illustrated by $2 \cdot \sigma_x$ bands drawn around the trajectory of the expected value. These diverge strongly initially due to the weak attraction of $\overline{x}_-^*$ but become narrower when $\overline{x}_+$ approaches the stationary solution. The constant 2σ bands from about $t = 12$ on correspond to the variance in equilibrium and it can be seen that these do nicely capture the remaining fluctuations around $\overline{x}_-^*$.

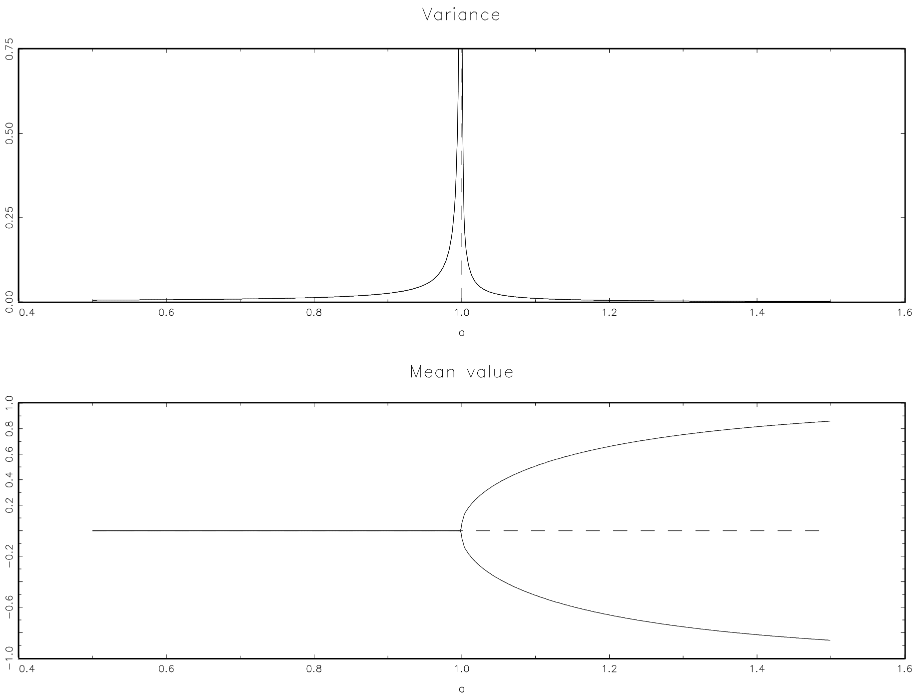

Figure 5: Dependency of mean values and variances in steady state on α.

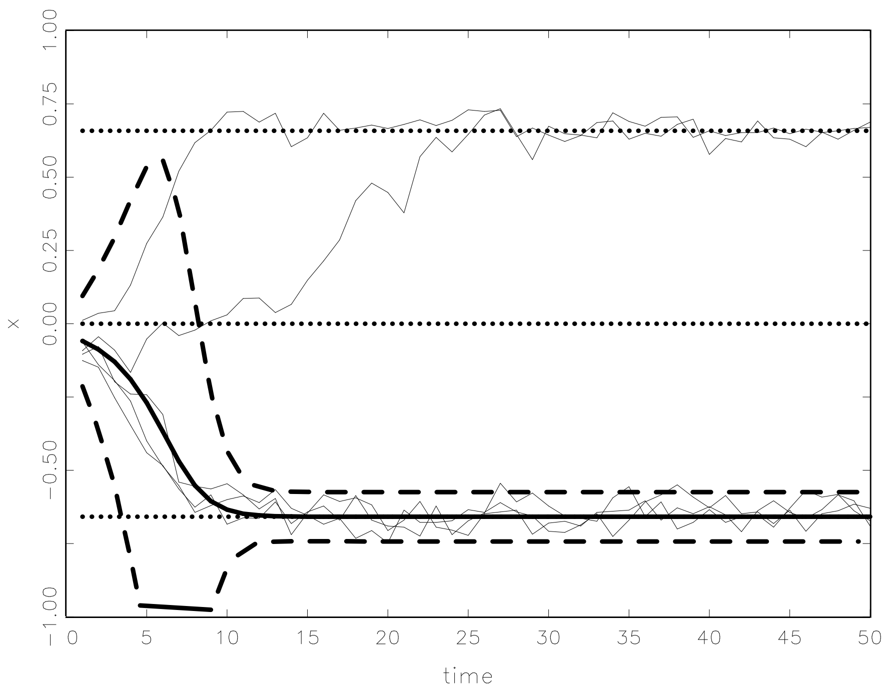

Figure 6: Transient development of first and second moments and microscopic simulations.

(3) An asset pricing model

In this application, we interpret the former "+" and "-" groups as bullish and bearish speculators, who are influenced by herd effects together with observed price changes. Their transition probabilities, therefore, include two terms:

$$p_{+-} = \nu exp(\alpha_1 x + \frac{\alpha_2}{\nu} p^{'}(t)), \quad (89)$$

$$p_{-+} = \nu exp(-\alpha_1 x - \frac{\alpha_2}{\nu} p'(t)), \quad (90)$$

where the price change $p^{'}(t)$ reinforces or weakens the herding tendency depending on whether its sign is in harmony or not with a bullish (bearish) attitude.[3] Following the lines of our previous derivations, we can establish the mean value dynamics for the opinion index for the average bullish or bearish market sentiment (which is pretty close in its structure to some published indices of investor sentiment):

$$\frac{d\bar{x}_t}{dt} = 2\nu\{tanh(\alpha_1 \bar{x}_t + \frac{\alpha_2}{\nu} p'(t)) - \bar{x}_t cosh(\alpha_1 \bar{x}_t + \frac{\alpha_2}{\nu} p'(t)) \quad (91)$$

In order to close the model, we have to add a hypothesis for price adjustment. A simple possibility is Walrasian price adjustment in reaction to excess demand (ED) with a certain adjustment speed β:

$$p'(t) = \frac{dp}{dt} = \beta ED. \quad (92)$$

Following a well-known strand of research in financial economics, excess demand in our financial market could be decomposed into two components: excess demand by chartists (ED_c) and excess demand by fundamentalist traders (ED_f).

The chartists might be just those whom we have classified as bullish or bearish in the agent-based component of the model. If chartists have a trading volume t_c this amounts to:

$$ED_c = (n_+ - n_-)t_c = 2Nxt_c = xT_c \quad \text{with} \quad T_c = 2Nt_c \quad (93)$$

[3]Division by ν of the second term is for technical reasons: An agent considers the price change during the mean time interval between switches between groups (which is ν^{-1}).

following the definition of the opinion index $x = \frac{n_+ - n_-}{2N}$. Fundamentalists, in contrast will have their excess demand depending on the difference between the perceived fundamental value P_f and the current market price:

$$ED_f = T_f(p_f - p), \tag{94}$$

with T_f the proportional trading volume of fundamentalists. Putting both components together, we arrive at the price adjustment equation:[4]

$$\frac{dp}{dt} = \beta(\overline{x}T_c + T_f(p_f - p)). \tag{95}$$

Eqs. (91) and (95) formalize our interdependent dynamic system in which the group dynamics influences the price dynamics and the price development feeds back on investor sentiment.
In studying the resulting system, we might first explore the question of existence and uniqueness or multiplicity of equilibria. Steady states of the joint opinion and price dynamics require $\frac{d\bar{x}_t}{dt} = \frac{dp}{dt} = 0$. Since this implies that the new second component of the herding probabilities is zero in any steady state, we arrive at the joint condition:

$$\frac{d\bar{x}_t}{dt} = \frac{dp}{dt} = 0 \Longrightarrow tanh(\alpha_1 \overline{x_t}) = \overline{x_t} \wedge p_t^* = \frac{T_c}{T_f}\overline{x} + p_f \tag{96}$$

Inspection reveals the following:

(i) for $\alpha_1 \leq 1$ we have a unique equilibrium $\overline{x_0}$ together with $p^* = p_f$.

(ii) for $\alpha > 1$ we encounter the two majority equilibria $\overline{x_+^*}$ and $\overline{x_-^*}$ (now bullish and bearish majorities) with pertinent prices $p_\pm^* = \frac{T_c}{T_f}\overline{x_\pm^*} + p_f$.

Hence, if herding is weak (case (i)) the price converges to the fundamental value (on average); if herding is strong (case(ii)), the equilibrium price comes along with an overvaluation or undervaluation of the asset compared to its fundamentals.

However, there are additional possibilities in this more complicated system: both $\overline{x_0^*}$ and the majority states $\overline{x_\pm^*}$ could be unstable (stability conditions are more involved than in the one-dimensional case). In such

[4] The price equation could in principle, also be formalized as a Poisson process with transition probabilities for price changes in upward and downward direction, cf. Lux (1997).

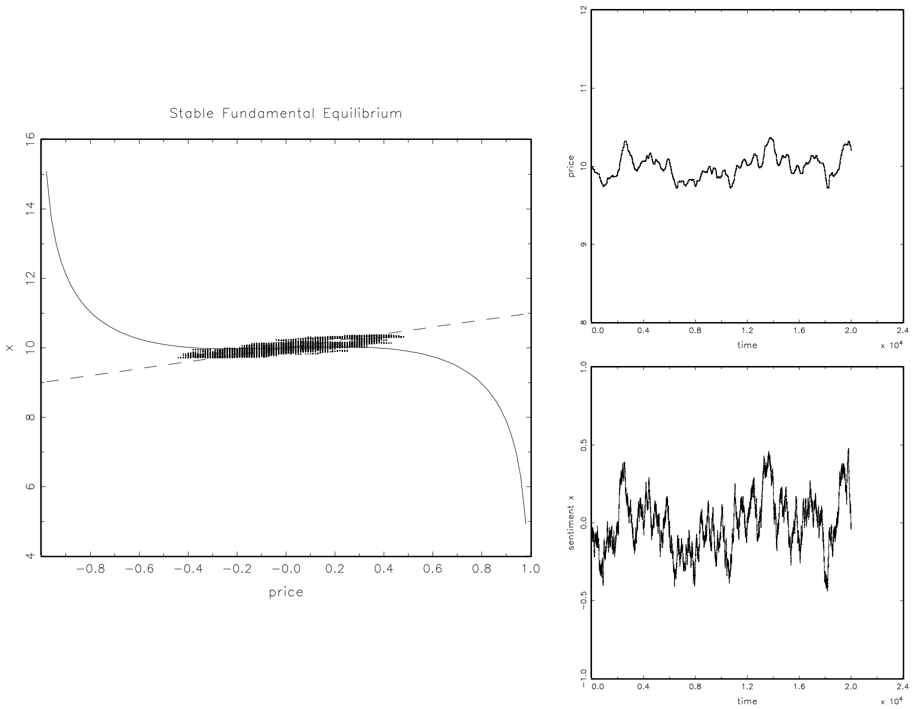

Figure 7: Stable fundamental equilibrium: minor unsystematic fluctuations around $(\overline{x^* = 0, p^* = p_f})$, parameters are $\nu = 0.5, \beta = 1, T_c = T_f = 0.5, p_f = 10, \alpha_1 = 0.8, \alpha_2 = 0.25, N = 100$. Left: simulation result as coordinates in (x, p) space, right: time series.

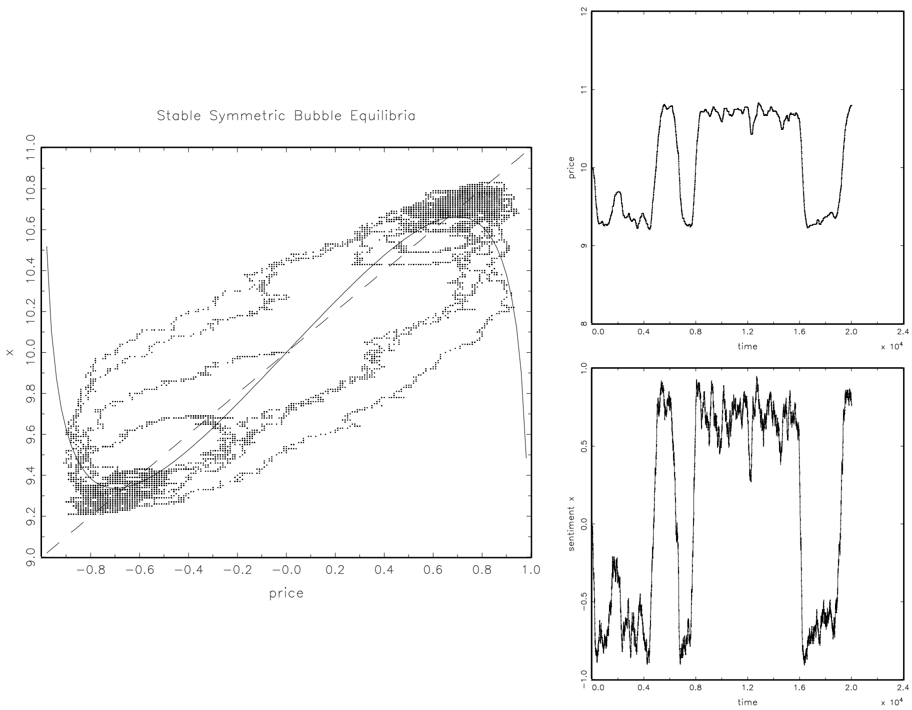

Figure 8: The case of symmetric "bubble" equilibria: the system tends towards $(\overline{x_+^*}, p_x^*)$ or $(\overline{x_-^*}, p_-^*)$ but also switches occasionally between phases with overvaluation and undervaluation. Parameters as before except for $\alpha_1 = 1.2, \alpha_2 = 0.75$.

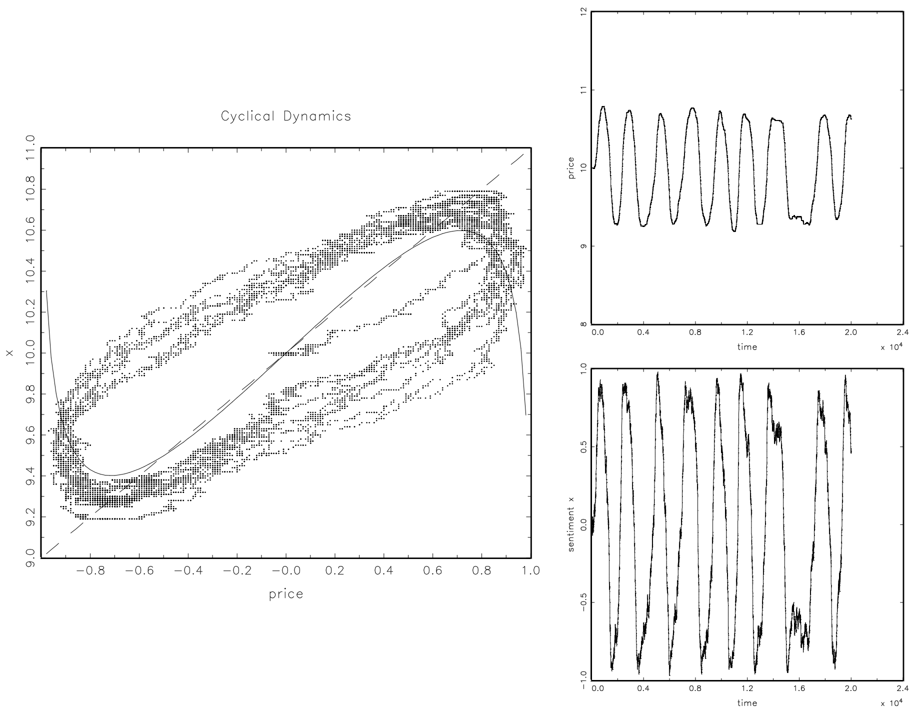

Figure 9: The case of cyclical variation between bullish and bearish phases. Parameters as before except for $\alpha_1 = 1.1, \alpha_2 = 0.95$

a scenario, the market performs almost regular cycles between overvaluation and undervaluation accompanied by investor sentiment oscillating between bullish and bearish majorities, cf. Fig. 9. Expanding our methodology above to the 2D case, we could also characterize the fluctuations in different market phases via the variance dynamics and the time development of covariance between p and $\overline{x}$, cf. Lux (1997)

The case of two interacting populations

Assume there are two groups with members $2N$ and $2M$ respectively and two opinions (strategies) "1" and "2" within each group:

$$2M = m_1 + m_2 \tag{97}$$

$$2N = n_1 + n_2 \tag{98}$$

The configuration of the overall population consists of the group occupation numbers $\{m_1, m_2, n_1, n_2\}$ or more compactly $\{m, n\}$ with $m = \frac{m_1 - m_2}{2}$, $n = \frac{n_1 - n_2}{2}$.

Movements between subgroups could depend on the distribution of "1" and "2" attitudes within the same population, but might also be influenced by the distribution of attitudes within the second group. Individual transition rates might then be written as:

$$p_{12}^{\mu} = V_{\mu}exp(\delta^{\mu} + \kappa^{\mu\mu}m + \kappa^{\mu\nu}n) = V_{\mu}exp(\Delta u^{\mu}(m,n)) \quad (99)$$

$$p_{21}^{\mu} = V_{\mu}exp(-\delta^{\mu} - \kappa^{\mu\mu}m - \kappa^{\mu\nu}n) = V_{\mu}exp(-\Delta u^{\mu}(m,n)) \quad (100)$$

$$p_{12}^{\nu} = V_{\nu}exp(\delta^{\nu} + \kappa^{\nu\mu}m + \kappa^{\nu\nu}n) = V_{\nu}exp(\Delta u^{\nu}(m,n)) \quad (101)$$

$$p_{21}^{\nu} = V_{\nu}exp(-\delta^{\nu} - \kappa^{\nu\mu}m - \kappa^{\nu\nu}n) = V_{\nu}exp(-\Delta u^{\nu}(m,n)) \quad (102)$$

If one follows all the previous steps, one derives the joint mean value equations for m and n:

$$\frac{dm}{dt} = 2V_{\mu}\{M\sinh(\Delta u^{\mu}(m,n)) - m\cosh(\Delta u^{\mu}(m,n))\} \quad (103)$$

$$\frac{dn}{dt} = 2V_{\nu}\{N\sinh(\Delta u^{\nu}(m,n)) - n\cosh(\Delta u^{\nu}(m,n))\} \quad (104)$$

The possibility of spillovers between groups allows for a rich variety of outcomes. Consider the simple version: $\delta^{\mu} = \delta^{\nu} = 0$, $\kappa^{\mu\mu} = \kappa^{\nu\nu} = \widetilde{\kappa}$, $M = N$, $V_{\mu} = V_{\nu}$, and define $\widetilde{\kappa}^{\mu\nu} = \kappa^{\mu\nu}M$, $\widetilde{\kappa}^{\nu\mu} = \kappa^{\nu\mu}N$. The opinion indices have dynamics:

$$\frac{d\overline{m}}{dt} = sinh(\widetilde{\kappa}\overline{m} + \widetilde{\kappa}^{\mu\nu}\overline{n}) - \overline{m}cosh(\widetilde{\kappa}\overline{m} + \widetilde{\kappa}^{\mu\nu}\overline{n}) \quad (105)$$

$$\frac{d\overline{m}}{dt} = sinh(\widetilde{\kappa}^{\nu\mu}\overline{m} + \widetilde{\kappa}\overline{n}) - \overline{n}cosh(\widetilde{\kappa}^{\nu\mu}\overline{m} + \widetilde{\kappa}\overline{n}) \quad (106)$$

The following scenarios can be found (cf. the subsequent illustrations):

1. Weak internal agglomeration within groups together with weak segregation tendencies between groups (e.g. $\widetilde{\kappa} = 0.2$, $\widetilde{\kappa}^{\mu\nu} = \widetilde{\kappa}^{\nu\mu} = -0.5$), the dynamics tends to a homogenous mixture:

$$(\overline{m}^*, \overline{n}^*) = (0,0). \quad (107)$$

2. With higher agglomeration tendencies within groups (e.g., $\widetilde{\kappa} = 0.5$) and higher segregation tendencies between groups (e.g. $\widetilde{\kappa}^{\mu\nu} = \widetilde{\kappa}^{\nu\mu} = -1$) a situation emerges where groups choose different alternatives.

3. For asymmetric inter-group dependency (e.g. $\widetilde{\kappa}^{\mu\nu} = -\widetilde{\kappa}^{\nu\mu} = -1$ one observes osciallatory convergence to a mixed population within both groups.

4. If the asymmetry becomes more pronounced (e.g. $\widetilde{\kappa}^{\mu\nu} = -1$, $\widetilde{\kappa}^{\nu\mu} = -10$, the divergent trends of self-separation from the behavior of the other group leads to permanent cycles.

Illustrations can be found in Figs. 10 to 14.

References:
The mathematical apparatus for the analysis of group dynamics and various applications can be found in:

Weidlich, W. and G. Haag (1983), *Concepts and Models of a Quantitative Sociology.*, Berlin.

Weidlich, W. (2002), *Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences*, London.

The financial market model is from:

Lux, T. (1995), Herd Behavior, Bubbles and Crashes, *Economic Journal* 105, 881–896

Lux, T.(1997), Time Variation of Second Moments from a Noise Trader/ Infection Model, *Journal of Economic Dynamics & Control* 22, 1–38

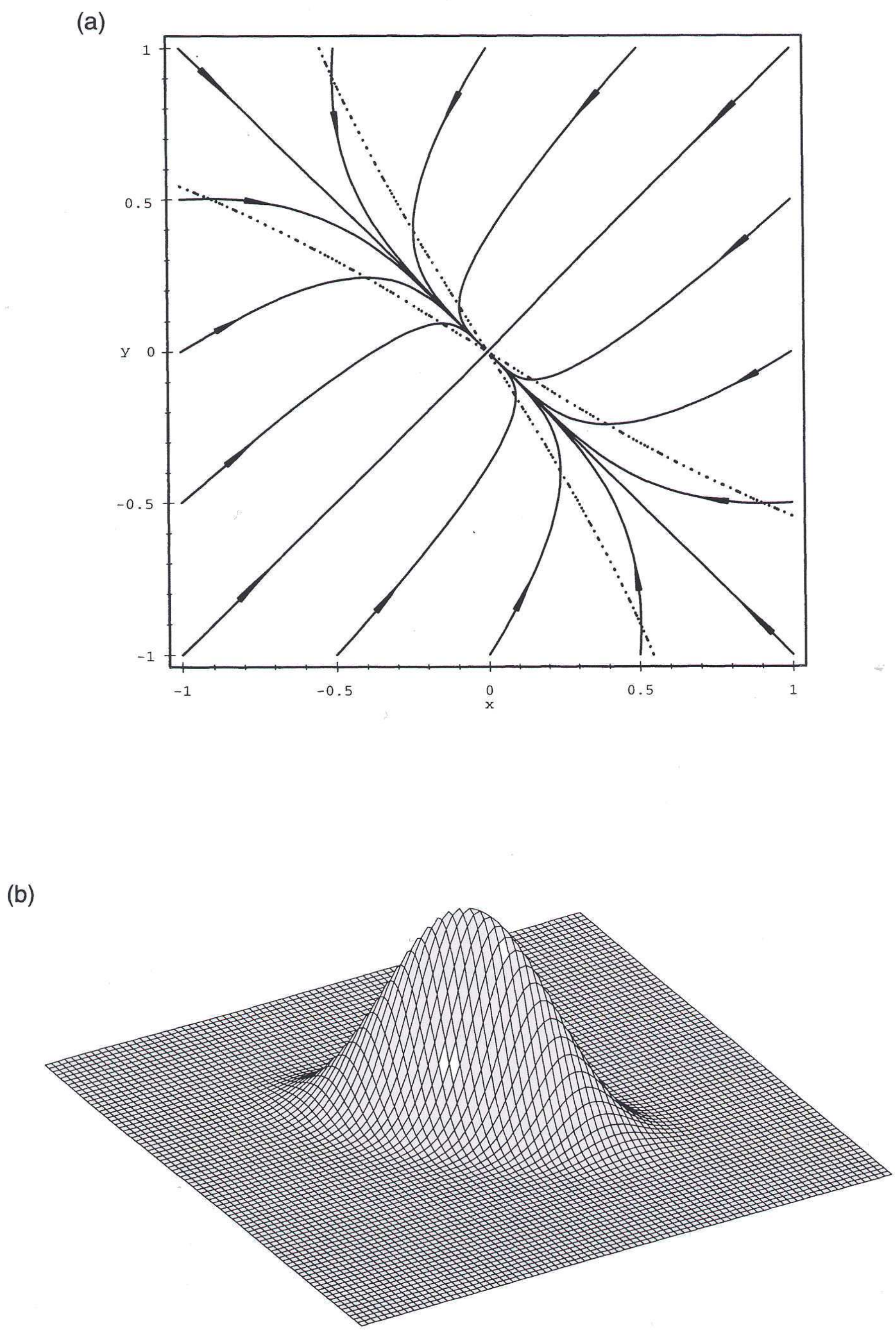

Figure 10: (a) Parameters $\widetilde{\kappa} = 0.2$ and $\widetilde{\kappa^{\nu\mu}} = 0.5$. Weak internal agglomeration trend and weak symmetrical reciprocal segregation trend. All fluxlines approach thr origin $(0,0)$ which describes the homogenous mixture of populations $\mathcal{P}^{\mu}$ and $\mathcal{P}^{\nu}$ and is the only stable stationary point; (b) Parameters as in Figure (a). $2N = 80$; Unimodal stationary propability distribution peaked around the stable origin $(0,0)$.
Source: Weidlich (2000), p. 90.

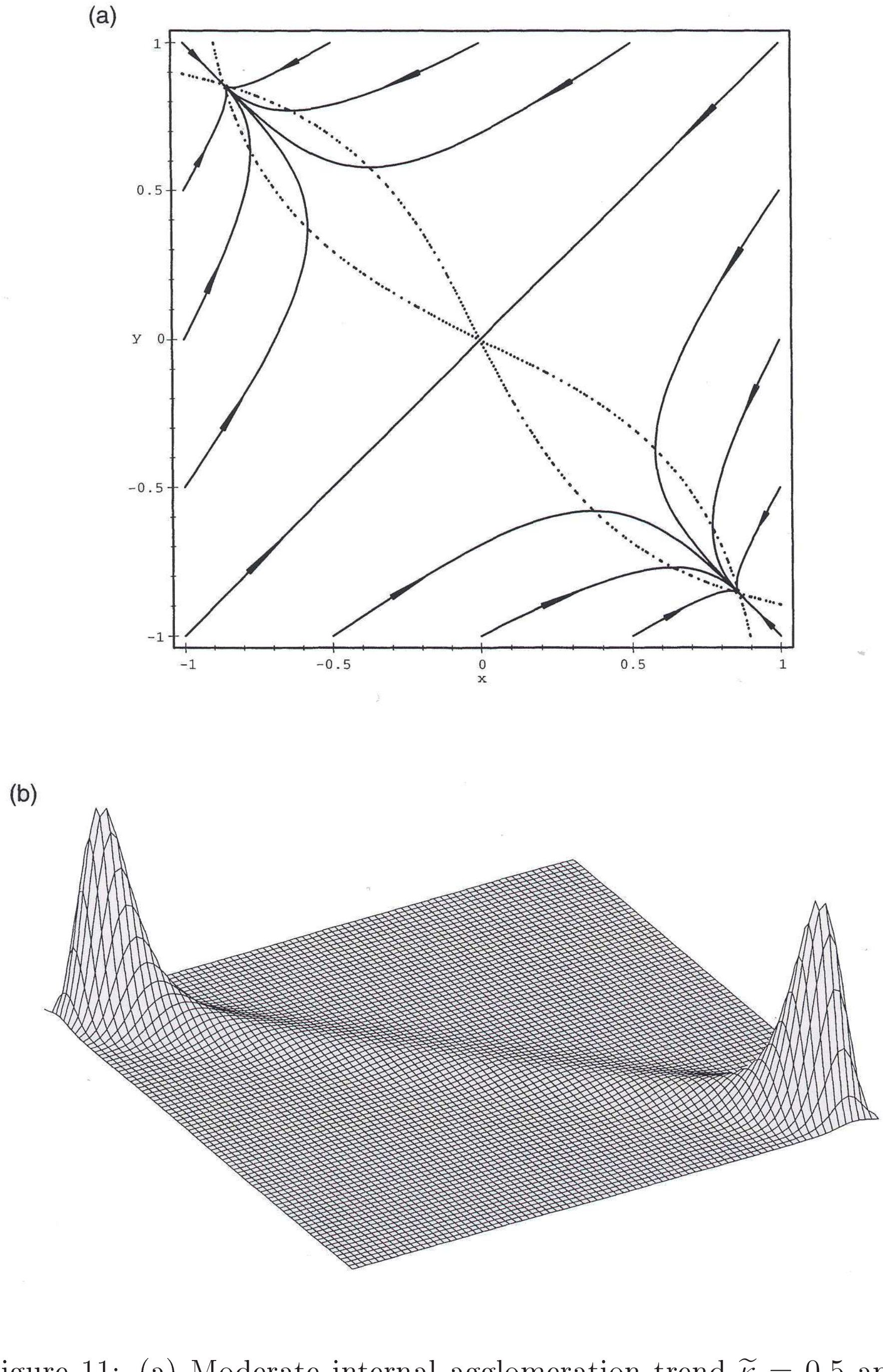

Figure 11: (a) Moderate internal agglomeration trend $\widetilde{\kappa} = 0.5$ and strong reciprocal segregation trend $\widetilde{\sigma} = 1.0$. The two stable stationary points in the second and fourth quadrant describe stable segregation of populations $\mathcal{P}^{\mu}$ and $\mathcal{P}^{\nu}$ in separate "ghettos". The fluxlines approach one of these stable equlibrium points; (b) Parameters as in Figure (a), $2N = 80$. The bimodal stationary probability distribution is peaked around the stationary points.
Source: Weidlich (2000), p. 92.

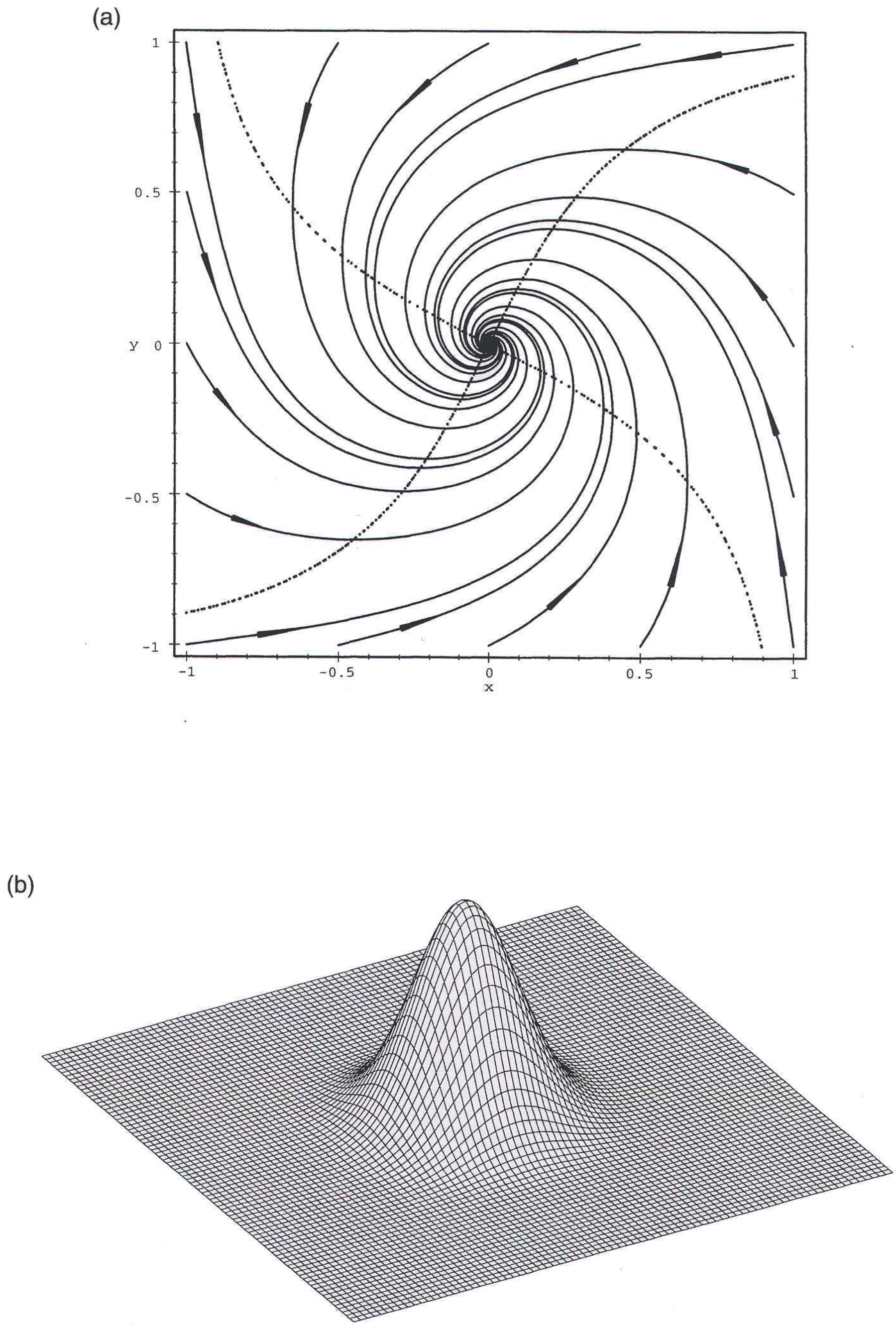

Figure 12: (a) Moderate internal agglomeration trend $\widetilde{\kappa}$ and strong asymmetric interaction $\widetilde{\kappa^{\mu\nu}} = -1.0$ and $\widetilde{\kappa^{\nu\mu}} = +1.0$. There exists one stable focus, the orgin (0,0) into which all fluxlines spiral; (b) Parameters as in (a), $2N = 80$. The unimodal stationary probability distributions is peaked around the stable focus (0,0).
Source: Weidlich (2000), p. 93.

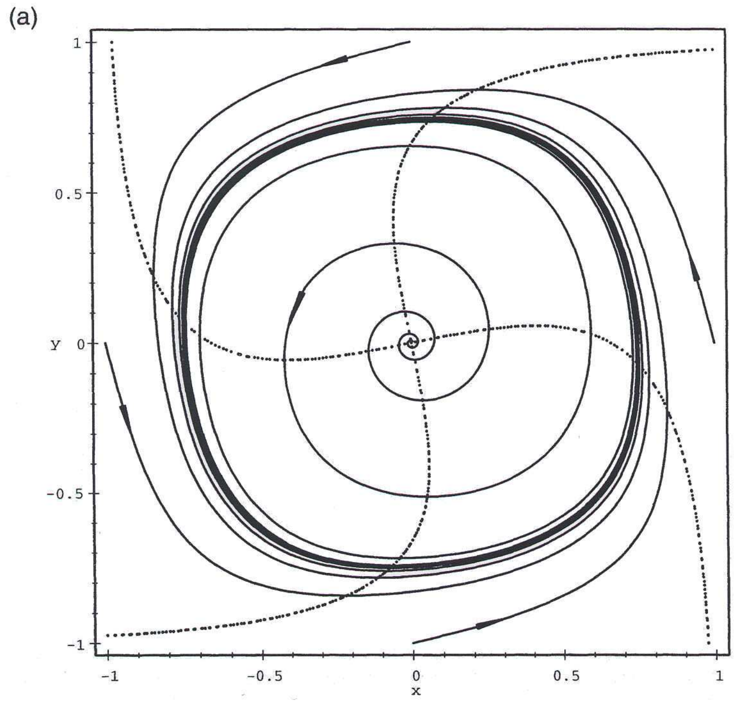

Figure 13: (a) Very strong internal agglomeration trend $\widetilde{\kappa} = 1.2$ and strong asymmetric interaction $\widetilde{\kappa^{\mu\nu}} = -1.0$ and $\widetilde{\kappa^{\nu\mu}} = +10$. The orgin (0,0) is an unstable focus. All fluxlines approach a limit cycle.
Source: Weidlich (2000), p. 94.

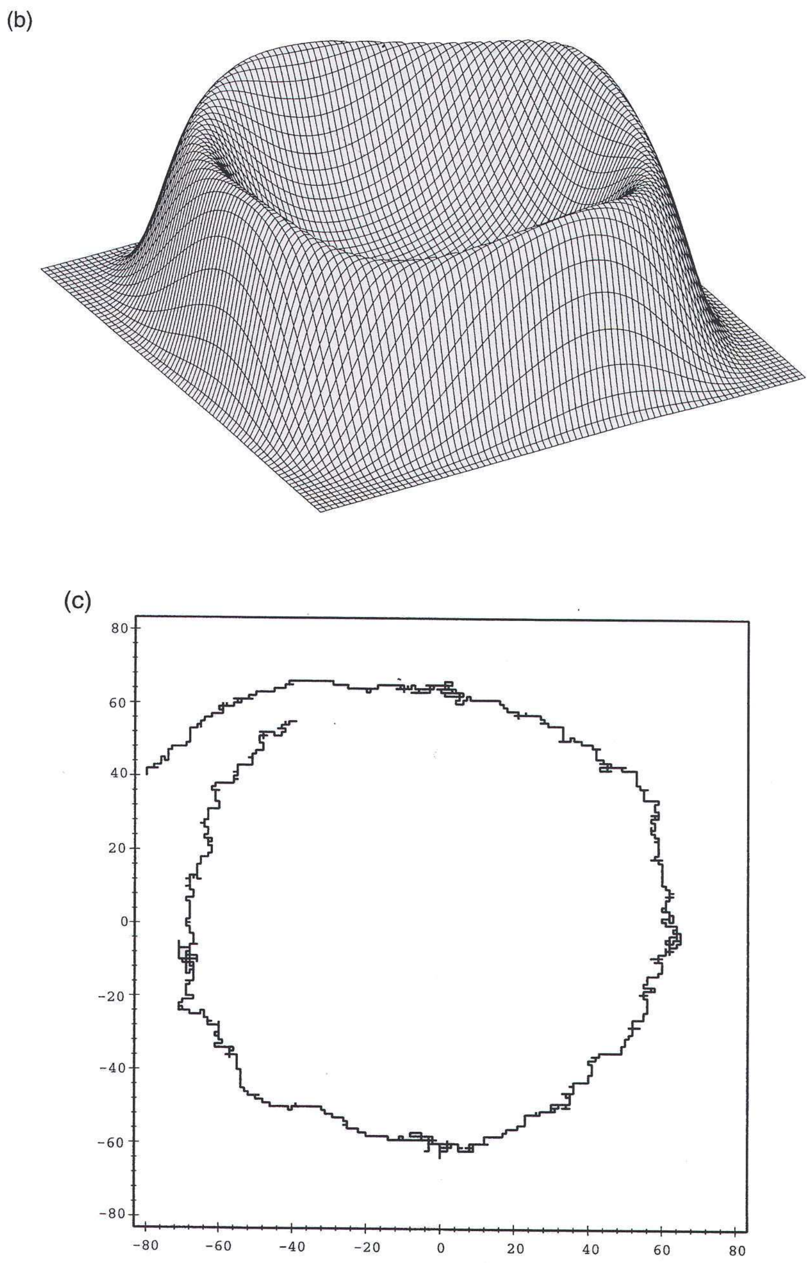

Figure 14: (b) Parameters in Fig.4(a), $2N = 80$. The quadrumodal stationary probability has four maxima corresponding to metastable situations and ridges between the maxima along the limit cycle; (c) Parameters as in Fig.4(a) and (b). example of stochastic trajectory belonging to transition rates. The trajectory abides around the metastable points of maximal probability and traverses at fast pace the states between the metastable situations.
Source: Weidlich (2000), p. 95.