Saving for the future: Don't leave it too late!

1. Introduction

Issues relating to personal finance appear both in the national news and in academic journals. The monthly announcement of the Bank of England base rates, consumer debt, house repossessions and pensions regularly hit the headlines.

One of the most important financial decisions facing young adults is planning for the future. Yet research conducted by the Aegon UK, a provider of life insurance and pension products, in summer 2007 found that 9.6 million people in the United Kingdom have no long-term savings plan or provision for a pension.[1] The Report talks of a 'reality gap' between what people expect later in life and what they are actually saving for.

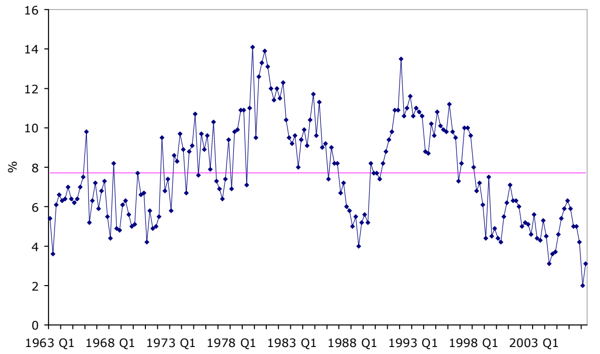
Of course, part of the reason for the 'reality gap' is that the future is exactly that, the future! But, it is probably true that many people are unable to perform basic financial calculations. Yet, it is possible to do the relevant calculations using an inexpensive scientific calculator in seconds. If one uses the properties of the exponential constant *e* it is a very easy task to make informed financial decisions. This is the objective of this paper.

2. How much are we saving?

In a textbook you will find the saving ratio defined as the proportion of *disposable income* that is not consumed. If you delve into the National Accounts you will find that the saving ratio is calculated as the proportion of *household resources* not consumed. This is because in preparing the National Accounts net contributions to both private and state pensions are deducted. But, if a household makes a net contribution to a private pension they are make a long-term savings decision. Hence, net contributions to private pension funds are added to disposable income to create an income measure known as household resources.

In the second quarter of 2007 the household resources measure was put at £224.03 billion. Of this £217.19 billion was consumed, leaving saving of £6.84 billion. This left the saving ratio at just 3.1%. Reference to Chart 1 shows this to be historically low; the historic low of 2% was actually in the preceding quarter. The average proportion of household resources saved between 1963Q1 and 2007Q1 is 7.7% (indicated on the chart).

Chart 1: Household savings ratio, %



Source: Table 2.5, Economic and Labour Market Review, Office for National Statistics

3. Effective interest rate

When financial products are advertised you will hear or see reference to the Annual Percentage Rate. This is the effective interest rate designed to allow consumers to compare more readily across financial products.

To see this consider the effective interest rate on a saving product *compounded n* times a year. If the yearly interest rate is *r*, it is credited as *r/n* on each of the *n* occasions. After 1 year the sum will have increased by the factor

(1) $(1 + r/n)^n$

[1] Further details can be found in the press release of 30 August 2007 available at http://www.aegon.co.uk/media/press_releases/pr20070830.htm

Hence, the APR or effective interest rate is

(2) $(1+r/n)^n - 1$

If a yearly rate of 6% is compounded every month the capital will increase over 1 year by a factor of 1.0617. This means the effective interest rate is 6.17%.

(3) $(1+0.06/12)^{12} = 1.0617$

If a sum *P* is invested for 1 year, with a yearly rate of interest *r* compounded *n* times, the final value will be

(4) $V = P(1+r/n)^n$

If, however, the sum is invested for 2 years with the same rate of interest and frequency of compounding, its final value will be

(5) $V = P(1+r/n)^n(1+r/n)^n$

We can write (5) as

(6) $V = P(1+r/n)^{2n}$

We can generalise the solution to (6) so as to find the final value of a lump sum *P* invested for *t* years with a yearly rate of interest *r* compounded *n* times. This will be

(7) $V = P(1+r/n)^{nt}$

4. Continuous compounding

Consider now the frequency with which interest is compounded. Interest could be compounded weekly, daily, or even more frequently. As the frequency of compounding increases, the effective interest increases, but ever more slowly. The theoretical extreme is known as *continuous compounding*.

While a theoretical nicety, continuous compounding has some neat implications for the mathematics of finance. To see this, we introduce the exponential constant. The exponential constant *e* is approximately 2.718281828. e^x can be defined at the limit where *n* approaches infinity as

(8) $e^x = (1+x/n)^n$

This is particularly relevant to the world of finance if one bears in mind that the effective interest rate or APR of a yearly rate *r* compounded *n* times a year is given by (2). It means that if we compound an annual rate *r* continuously the APR is given by

(9) $e^r - 1$.

If the annual rate is 6% then the APR is 6.184%.

(10) $e^{0.06} - 1 = 0.06184 = 6.184\%$

We can rewrite the final value of a lump sum invested for *t* years under continuous compounding as

(11) $V = Pe^{rt}$

5. Definite integration and savings plans

So far we have applied continuous compounding and the properties of the exponential constant to a lump sum investment. But, in planning for the future or for retirement individuals may consider putting in a place a *savings plan* to which they make regular contributions.

We will consider savings plans where *£A* is paid continuously during the year. By assuming continuous compounding we show how to employ a technique known as *definite integration* using the exponential constant. But, firstly we consider the properties of the exponential constant.

The slope of the graph of an exponential function is the same as the value of the function at that point. The slope of a graph of a function is called the *derivative of the function*. The derivative is a measurement of how a function changes when the values of its inputs change.

The process of finding a derivative is called differentiation. When one differentiates e^x one obtains e^x. Hence, symbolically, if $y = e^x$ the derived function is

(12) $\frac{dy}{dx} = e^x$

In other words, the function is unchanged by differentiating it.

Applying the chain rule we also have the result that if $y = e^{rx}$ then the derived function is

(13) $\frac{dy}{dx} = re^{rx}$

Integration is the process of identifying the function that differentiates to a derived function. More simply, it is the reverse of differentiation. Therefore, if one integrates e^x the function is also unchanged. Hence, symbolically,

(14) $\int e^x dx = e^x$

To integrate e^{rx} we must also do the reverse of differentiation so we now divide by *r* rather than multiply so

(15) $\int e^{rx} dx = \frac{e^{rx}}{r}$

From the property of any index, $e^0 = 1$. The reason this is important for problems involving time is that almost always the start time is designated time 0, whatever the actual start year.

We are now in a position to evaluate a *definite integral* with 0 as the lower limit. This can be used to calculate the *factor* by which we multiply the annual *£A* paid continuously into a savings plan so as to determine its final value.

Assume an interest rate of *r*, which is compounded continuously, and an upper limit for our integral *T*, which is the length of the saving plan.

(16) $\int_0^T e^{rx} dx = \left[\frac{e^{rx}}{r} \right]_0^T$

Solving (16) gives

(17) $\frac{e^{r*T}}{r} - \frac{e^{r*0}}{r}$

Since $e^{r*0} = e^0 = 1$, this simplifies to

(18) $\frac{e^{r*T} - 1}{r}$

To illustrate, assume that the interest rate, *r*, is 0.06 and that the savings plan runs for 15 years. This is the upper limit of the definite integral. We now substitute the values of *r* and *T* into (18)

(19) $\frac{e^{0.06*15} - 1}{0.06}$ = 24.33 to 2dp

The value of a plan where the annual amount saved is *£A*, will be worth nearly 24 times *£A* after 15 years.

If an annual value of £1,200 is saved continuously for 15 years at an interest rate of 6% is £1,000, then the final value of the savings plan can be determined by the definite integral

(20) $V = £1{,}200 \int_0^{15} e^{0.06x} dx$

In solving (20) we find the final value of the plan V after 15 years is

(21) $V = £1{,}200 \left[\frac{e^{0.06*15} - 1}{0.06} \right] = £29{,}192$

Savings plans never actually involve continuous compounding or accumulation. Yet, while theoretical niceties, they enable us to undertake financial calculations on a calculator in seconds. Importantly, they provide for excellent approximations to more 'realistic' financial calculations. We can show this by slightly amending our current example.

Assume our individual saving £100 per month for 15 years at a constant interest rate of 6% p.a. compounded monthly. This 'realistic' plan has 180 monthly payments with interest credited as 0.06/12 each month. The final value of the plan can be written as

(22) $100(1.005) + 100(1.005)^2 + ... + 100(1.05)^{180}$

We now have a *geometric progression*. Each term is the previous term multiplied by *1.005*. This is the *common ratio*. The first term in the progression, *100*1.005*, is the *scalar*. The summation of the *n* terms, S_n, can be found by applying the following general formula, where *b* is the scalar and *y* the common ratio

(23) $S_n = \frac{b(1-y^n)}{1-y}$

If we now substitute in for *b* and *y*, we find that the value of (22) is equal to £29,227 to the nearest £.

(24) $\frac{100\,(1.005)(1-1.005^{180})}{1-1.005} = £29{,}227$

The difference between the estimated final values is £35. When expressed *relative* to the final value of the 'realistic' plan, the difference amounts to (35/29,277)*100 or just 0.12%.

6. Planning for the future

The Pensions Service[2] provides information about pensions and, in the current context, planning ahead for the future. In addressing the question as to when to start planning for retirement, the Pensions Service offers the following advice:

> *Like many other people you might feel that retirement is too far away to think about. Maybe you think you can't afford to save for the future when there are so many other things you have to pay for every day. But the truth is the earlier you start saving for retirement the more money you're likely to have to enjoy yourself.*[3]

Is this sound advice or simply scaremongering? A simple example amply demonstrates the relevance of the Pension Service's advice.

Consider a 23 year old who has just left university and plans to save £1,000 per year for 45 years to provide for her pension needs. Assume the return is estimated at 7% per year. The final value of this savings plan can be determined by the definite integral

(25) $V = £1{,}000\int_0^{45} e^{0.07x}dx$

[2] The Pensions Service is part of the Department for Work and Pensions. Its website address is http://www.thepensionservice.gov.uk/

[3] http://www.thepensionservice.gov.uk/planningahead/start/when-should-i-save.asp

Solving this we find that the final value of the plan for our individual, at which time she will be 68 years old, is £319,087.

(26) $£1{,}000\left[\frac{e^{0.07*45}-1}{0.07}\right] = £1{,}000*319.087$

Now suppose that due to cash flow problems this person delays saving for 15 years. Consider the impact this has on her yearly payment in order to deliver the same lump sum of £319,087 when she is 68 years old.

In our formula we know *V* = £319,087 but *T* is now 30 since she will only save for 30 years. Therefore,

(27) $£319{,}087 = A\int_0^{30} e^{0.07x}dx$

We now solve for the definite integral on the RHS of (27) and make *A* the subject of the equation.

(28) $A = \frac{£319{,}087}{102.3739} = £3{,}117$

The annual amount that would need to be saved is £3,117. By delaying saving our individual pays £3,117 for 30 years compared to £1,000 for 45 years so as to have the same final value. This amounts to an additional £48,510.[4]

This is of course a simplified example. In reality pension contributions are seldom constant over time. Nonetheless, it goes to demonstrate that the Pensions Service advice to start saving early is sound advice indeed!

Tasks

Assume an individual saves £500 per year continuously for 30 years at an interest rate of 7.5%.

(i) What is the final value of the plan?

(ii) How much must the individual save each year in an equivalent plan of 20 years so that it has the same final value?

(iii) Compare the total payments made in the 30 and 20 year plans.

[4] The total payment in the 45 year plan is £45,000, but £93,510 in the in the 30 year plan.