Affording the mortgage

1. Introduction

Economists are not financial advisers, but there are few who have not been approached by friends wanting to talk about their financial situation. This is especially true of mortgages. Though not financial advisers, economists are nonetheless interested in the decisions that individuals make in relation to housing.

In this case study we will focus on the repayment costs of a mortgage. A mortgage is a loan secured on a property. The most common type of mortgage is a *repayment mortgage* where payments are made to cover interest costs and repay the capital. These payments are known as the costs of servicing the debt. Figures for 2007 from the Council of Mortgage Lenders (CML)[1], the trade body that represents lenders, show that 67% of first-time buyers opted for a repayment mortgage.

Some borrowers opt for an interest-*only mortgage*, but with a specified repayment vehicle, such as a savings plan, designed to cover the repayment of capital at some point. However, the CML's statistics show that in 2007, 20% of first-time buyers took out an interest-only mortgage without any specified repayment vehicle.

We demonstrate two distinct methods for calculating the annual mortgage costs of a repayment mortgage. The first approach is to treat each time period as discrete. The amount owing at the end of each period is calculated. The annual repayment is sufficient that at the end of the mortgage the balance outstanding is zero.

The second approach to calculating the annual repayment applies the concept of present value concept. We treat the value of the loan as money received *today*. This is then equated with the present value of repayments which occur in the *future*.

[1] The homepage of the Council of Mortgage Lenders can be accessed at http://www.cml.org.uk

2. Discrete approach

To determine the annual repayment we can model the amount owing on a repayment mortgage at the end of a particular year. The approach can be easily adapted so as to calculate the amount owing at the end of time periods of any given length. In our modelling we will use letters to represent our variables. The size of the original mortgage, also known as the advance, will be represented by *A*, the amount owing by *D* and the annual rate of interest by *r*.

We will assume that interest is calculated by the lender on annual basis. In practice, this is rarely the case. But, again the method can be adapted to cater for this. By calculating interest annually, households making monthly payments remain liable to pay interest on the amount outstanding at the beginning of the year. The benefits of the repayments therefore do not accrue to the following year. We will represent the annual repayment by *m*. The amount owing, *D*, on an advance, *A*, at the end of year 1 can be written as

(1) $D_1 = A(1+r) - m$

At the end of the second year, the amount the household owes will be the balance at the start of the year, as determined by equation (1), *plus* the interest charged on this balance, but *minus* the annual repayment. This can be written as

(2) $D_2 = [A(1+r) - m](1+r) - m$

If we multiply equation (2) out, this is equivalent to

(3) $D_2 = A(1+r)^2 - m(1+r) - m$

You might be able to see that a pattern is emerging on the RHS of our equations. This becomes even clear if we consider the amount owing at the end of the third year. We apply interest to the debt brought forward from year 2 and subtract the annual repayment.

(4) $D_3 = [A(1+r)^2 - m(1+r) - m](1+r) - m$

This is equivalent to,

(5) $D_3 = A(1+r)^3 - m(1+r)^2 - m(1+r) - m$

A pattern on the RHS is now established and we can write an expression for the amount owing at the end of any particular year. Consider year *n*, when the mortgage is finally paid off. For many mortgages we would expect this to be after 25 years. If we denote the mortgage outstanding at its completion as D_n, we can set it equal to zero,

(6) $D_n = 0$

Using the pattern we have seen emerging, we can write this as

(7)

$$0 = A(1+r)^n - m(1+r)^{n-1} - m(1+r)^{n-2} - ... - m(1+r) - m$$

If we take all but the first term on the RHS of (7) over to the LHS we get the expression

(8)

$$m + m(1+r) + ... + m(1+r)^{n-2} + m(1+r)^{n-1} = A(1+r)^n$$

The LHS is a series known as a *geometric progression*. This is because each term is the previous term multiplied by 1+*r*. The annual amount repaid, *m*, is known as the *scalar*, while *1+r* is the *common ratio*. The summation of n terms of a geometric progression, S_n, can be found by applying the following general formula, where *b* is the scalar and *y* the common ratio

(9) $S_n = \frac{b(1-y^n)}{1-y}$

The number of terms on the LHS will be the same as the numbers of years over which the mortgage runs. For example, for a 25 year mortgage we have 25 terms on the LHS. If we now substitute in for *b* and *y*, we find that the LHS of equation (8) is equal to

(10) $\frac{m(1-(1+r)^n)}{1-(1+r)}$

The denominator is equal to –*r*. If we multiply the top and bottom through by -1, then we can see that the LHS can also be written as

(11) $\frac{m((1+r)^n - 1)}{r}$

Having solved our geometric progression we can now write (8) as

(12) $\frac{m((1+r)^n - 1)}{r} = A(1+r)^n$

Finally, solving for *m*, we can determine the total repayment made each year

(13) $m = \frac{Ar(1+r)^n}{(1+r)^n - 1}$

To illustrate the use of the formula, consider the affordability of an average mortgage taken out in the United Kingdom in 2007. The CML reports 1,015,900 advances for house purchase over the year. This compares with 1,211,000 in 2006. Hence, there were 9.4% fewer loans advanced in 2007. The total value of loans advanced for house purchase in 2007 was £154.893 billion a fall of 1.6% from 2006 when their value totalled £157.386 billion. The median advance in 2007 was £127,042, up from £118,536 in 2006, while the median age of the house purchaser was unchanged at 35.[2]

The CML also reports that the average building society mortgage rate for 2007 was 5.69%.[3] If we take the case of a 25 year mortgage with interest applied annually, then the values to enter are *A*=£127,042, *r*=0.0569 and *n*=25.

(14) $m = \frac{£127{,}042 * 0.0569(1+0.0569)^{25}}{(1+0.0569)^{25} - 1}$

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[2] A median is used to remove the potential distortion from observations whose values are either very small or very large.

[3] The interest rate quoted is the end of year figure.

The annual repayment is found to be £9,647.24.

(15) $m = \frac{£28{,}834.1422}{2.9888} = £9{,}647.24$

This is equivalent to £803.94 per month. If our borrower was to remain on this particular mortgage product for 25 years they would pay £241,181. This means that will pay interest of £114,139, an amount equivalent in itself to 90% of the original amount borrowed[4].

3. Present value approach

Another way of modelling the annual repayment is to apply the present value concept. The advance is money received today. The repayments occur in the future. We can determine the annual repayment by equating the *present value* of the future repayments with the value of the advance.

In using the present value approach we will make use of the idea of *continuous compounding*. Compounding relates to the frequency with which interest is calculated. The calculation of interest is increasingly done on a more frequent basis, for instance, monthly or even daily. If interest is calculated *n* times a year with a yearly interest rate *r*, it is applied at *r/n* on each of the *n* occasions. The Annual Percentage Rate (APR) or effective interest rate becomes

(16) $(1 + r/n)^n - 1$

The discount factor is

(17) $(1 + r/n)^n$

If interest is calculated annually, such that *n* is 1, then the APR and the yearly interest rate, *r*, are the same.

As the frequency with which interest is calculated increases, the effective interest rate increases, but ever more slowly. The theoretical extreme is known as *continuous compounding*.

Consider now the exponential constant. The exponential constant *e* is approximately 2.718281828. e^x can be defined at the limit where *n* approaches infinity as

(18) $e^x = (1 + x/n)^n$

Since the APR of a yearly rate *r* compounded *n* times a year is given by (16), if we compound an annual rate *r* continuously the APR is given by

(19) $e^r - 1$.

Hence, the discount factor is

(20) e^r

If the annual rate is 5.69% then the APR under continuous compounding is 5.85%. The discount factor is 1.0585, meaning that the value of a lump sum, *P*, subject to this effective interest rate increases by the factor 1.0585 over 1 year. Over *t* years under continuous compounding a lump sum *P* would be worth

(21) $S = Pe^{rt}$

If we reverse the process, the present value *P* of a single final value *S* in *t* years time can be found by dividing the final value by e^{rt}, such that

(22) $P = \frac{S}{e^{rt}} = Se^{-rt}$

We can use the idea of continuous compounding to estimate annual mortgage repayments extremely quickly and accurately on a calculator. To do so one can think of mortgage repayments as being akin to a savings plan. We will assume that *m* is paid continuously during the year. We then employ a technique known as *definite integration* using the exponential constant.[5]

The slope of the graph of an exponential function is the same as the value of the function at that point. Hence, when we differentiate e^x we obtain e^x. Hence, symbolically, if $y = e^x$ the derived function is

(23) $\frac{dy}{dx} = e^x$

4 =(£114,139/£127,042)*100

5 This technique is also employed in the case study '*Saving for the future: Don't leave it too late*'.

In other words, the function is unchanged by differentiating it. Applying the chain rule we also have the result that if $y = e^{rx}$. The derived function is

(24) $$\frac{dy}{dx} = re^{rx}$$

Integration identifies the function that differentiates to a derived function and so is the reverse of differentiation. If we integrate e^x the function is also unchanged. Hence, symbolically,

(25) $$\int e^x dx = e^x$$

Similarly, if we integrate e^{-rx}, we find

(26) $$\int e^{-rx} dx = \frac{e^{-rx}}{-r}$$

To find the necessary annual repayment *m* we need to evaluate a definite integral. Specifically, we require that the *present value* of our stream of repayments (*m*) be equal to the initial advance (*A*).

(27) $$A = m\int_0^T e^{-rx}\,dx$$

Assume an interest rate *r*, which is compounded continuously, and an upper limit for our integral *T*, which is the length of the mortgage.

(28) $$A = m\int_0^T e^{-rx}\,dx = m\left[\frac{e^{-rx}}{-r}\right]_0^T$$

Solving (28) gives

(29) $$A = m\left[\frac{e^{-r*T}}{-r} - \frac{e^{-r*0}}{-r}\right]$$

Since $e^{r*0} = e^0 = 1$, this simplifies to

(30) $$A = -\frac{m}{r}(e^{-rt} - 1)$$

And, hence, the annual repayment is

(31) $$m = -\frac{Ar}{(e^{-rt} - 1)}$$

If we now substitute in the values for the average advance and mortgage rate in 2007 we find the yearly repayment over 25 years is £9,525.35.

(32) $$m = -\frac{£127{,}042 * 0.0569}{(e^{-0.0569*25} - 1)} = £9{,}525.35$$

This compares with £9,647.24 under the discrete approach with interest applied annually. Therefore, the effect of a slighter higher interest rate resulting from continuous compounding is outweighed by the fact that the value of the loan is being reduced during the year. The monthly repayment under continuous compounding amounts to £793.78. The total repaid over the 25 years is £ 238,134. The borrower therefore pays £111,092 in interest which is equivalent to 87% of the advance.[6]

4. Affordability

One measure of the affordability of mortgage payments is the proportion of gross income that a borrower-household must devote to the payments. The CML's reports the median gross income of the borrower-household in 2007 as £41,159. Annual debt servicing costs of £9,525 under continuous compounding and continuous payments equates to 23.1%[7] of gross income.

The proportion of income devoted to mortgage payments has implications for the sustainability of a household's investment in housing. The affordability of housing is not just about getting on the property ladder or indeed being able move up to the next rung, it is also about being able to safely remain there.

Tasks

(i) Calculate the annual cost of a 25 year repayment mortgage of £100,000, where interest is applied to the loan outstanding at the beginning of each year and the mortgage rate is 10% per annum. What is the total amount repaid by the borrower over the 25 years? What is the total interest paid by the borrower?

(ii) Repeat (i) but assuming continuous compounding and payments. What is the percentage difference between the sums repaid under both methods?

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[6] =(£111,092/£127,042)

[7] =(£9,525/£41,159)*100