Calculus of Variations, singular case: viscosity solutions

Saint Andrews February 26 2009

Introduction

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$$J[x(.)] := \int_0^\infty e^{-\delta t} \left[A(x(t)) + B(x(t))\dot{x}(t)\right] dt$$

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- $(\mathbf{H_5})$ $0 < C(a^+) := \lim_{x \to a^+} C(x)$, $0 > C(b^-) := \lim_{x \to b^-} C(x)$.

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- From $f^+(a) > f^-(a)$, $MRAP(a, \bar{x})$ exists.
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$$V(\cdot) = \frac{A(x_0) + V_2(\cdot)}{\delta}\,.$$

Continuity of $V_2(\cdot)$

Proposition Assume :

- (H_2) $f^{\pm}(.)$ lipschitz on $[a, b]$
- (H_5) $C(a+) > 0, \; C(b-) < 0$
- (H_6) $C(.)f^{\pm}$ lipschitz on (a, b)

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$\leq 2\varepsilon + |a-z| M \quad$ (Gronwall).

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$V_2(\cdot)$ is a viscosity solution on (a, b)

of the Hamilton-Jacobi :

$$\delta Z(x) \; - \; max[(C(x) + Z'(x))f^{-}(x), (C(x) + Z'(x))f^{+}(x)] \; = 0$$

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$$V_2(x) = \sup_{Adm(x)} \int_0^T e^{-\delta t} C(x(t))\dot{x}(t)dt \; + \; e^{-\delta T} V_2(x(T))$$

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$V_2(.)$ viscosity solution .

Value along MRAP

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f^{+} : $x < \bar{x}$; $\quad f^{-}$: $x > \bar{x}$

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Property :
1) $J_2[MRAP(x, \bar{x})](.)$ continuous w.r.t. x on (a, b).

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$$T_2(x) := \max_{\bar{x} \in \bar{X}} J_2[MRAP(x, \bar{x})]$$

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$J_2[MRAP(x, \bar{x})](.)$ continuous w.r.t. x on $[a, b]$

$$T_2(x) := \max_{\bar{x} \in \bar{X}} J_2[MRAP(x, \bar{x})]$$

From $f^+(a) > f^-(a) = 0$, $0 = f^+(b) > f^-(b)$,
then $T_2(.)$ continuous on $[a, b]$.

Value along a MRAP...

Second Main Result

$T_2(.)$ is a viscosity solution of

$$H(x, Z(x), Z'(x)) = \\ \delta Z(x) \; - \; max[(C(x) + Z'(x))f^{-}(x), (C(x) + Z'(x))f^{+}(x)] \; = 0$$

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3) $x < \bar{x}$,

$$H(x, T_2(x), T_2'(x)) = \delta T_2(x) - max[\frac{\delta}{f^+(x)} T_2(x) f^-(x), \delta T_2(x)]$$

$V_2(.)$ *and* $T_2(.)$

Proposition

Under assumptions $(H1)$-$(H5)$, we have :

- Under $f^+(a) > f^-(a)$, then $T(a) = V(a)$.
- Under $f^-(b) < f^+(b)$, then $T(b) = V(b)$.

Theorem (Bardi, Barles, ..)

Uniqueness of the solutions of H-J

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- $S(\cdot)$ and $W(\cdot)$ take same values on the boundaries $\partial\Omega$.
- $H(x, p)$ satisfy

$$|H(x,p) - H(y,p)| \leq F(|x-y|(1+|p|))$$

where $F : [0, \infty) \to [0, \infty)$ is continuous nondecreasing with $F(0) = 0$, for all $x, y \in (a, b)$ and $p \in R$.

Then $S(.) = W(.)$ on $\bar{\Omega}$

Optimality of the MRAPs

Therefore

$$V_2(.) = T_2(.), \ \ \forall x \in [a, b]$$

and thus

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We proved

The MRAPs are optimal solutions.