Stochastic Differential Equations, Part I

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Overview

- $\triangleright$ Motivation: Population dynamics
- $\triangleright$ Brownian Motion and Itô integral
- $\triangleright$ Stochastic differential equations
- $\triangleright$ Stratonovich integral and SDEs
- $\triangleright$ Stability

Motivation: Population dynamics

Verhulst model, logistic growth for single species

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(K - N(t)) \qquad , \; t \in [0,T], \; \text{i.v.}\, N_0$$

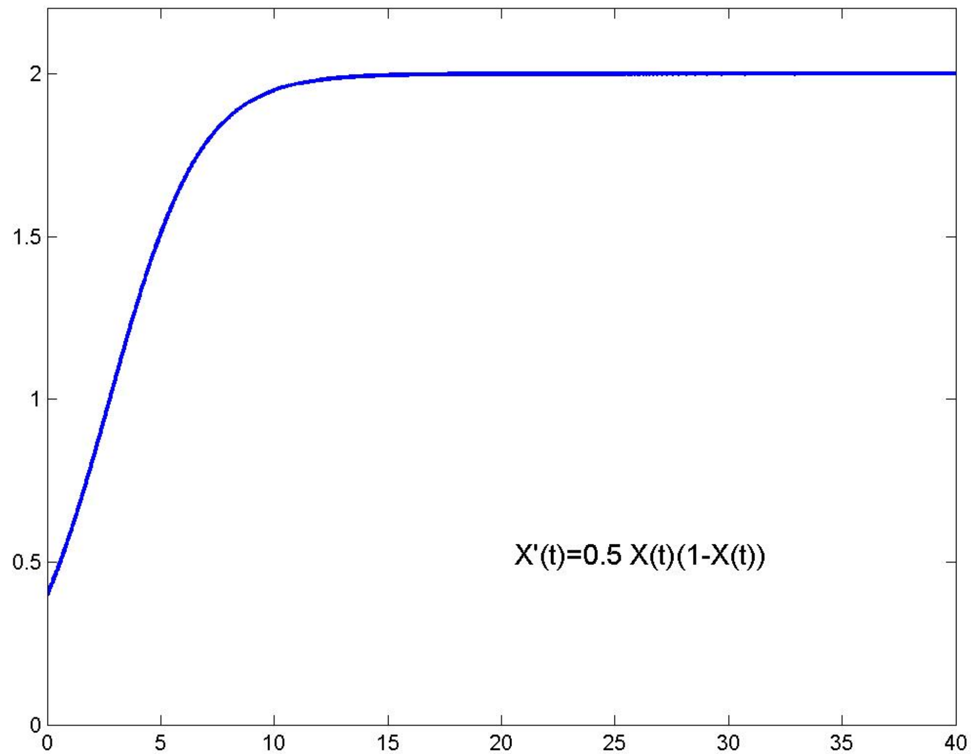
r growth parameter,

K carrying capacity

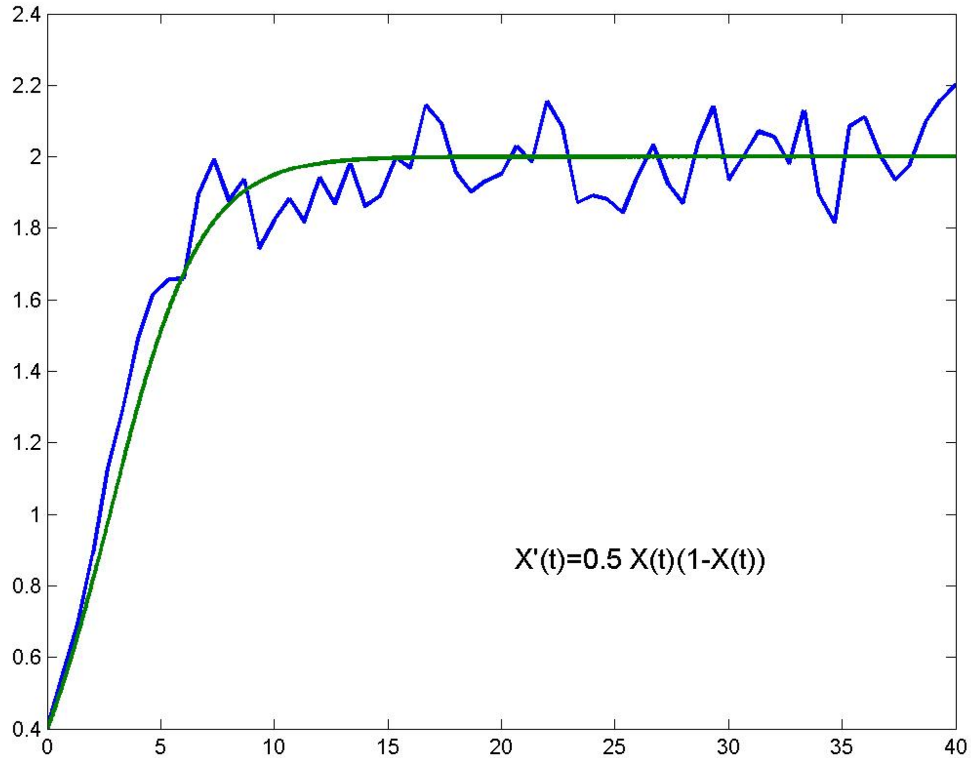
$r\,(K - N(t))$ birth per capita rate

Solution: $N(t) = \dfrac{K}{1 + K\,e^{-rKt}}$

Motivation: Population dynamics



Motivation: Population dynamics



Motivation: Population dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(K - N(t)) \qquad , \ t \in [0,T], \ \text{i.v.}\ N_0$$

set $K = \bar{K} +$ 'noise'

average + random variation to account for changes in the environment

Motivation: Population dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(\bar{K} - N(t)) + r\,N(t) \cdot \text{'noise'},\ t \in [0,T],\ \text{i.v.}\, N_0$$

Motivation: Population dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(\bar{K} - N(t)) + r\,N(t) \cdot \text{'noise'},\ t \in [0,T],\ \text{i.v.}\ N_0$$

How is the 'noise' modeled?

Motivation: Population dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(\bar{K} - N(t)) + r\,N(t)\cdot \text{'noise'},\ t \in [0,T],\ \text{i.v.}\, N_0$$

How is the 'noise' modeled?

What is the mathematical definition of such an equation?

Motivation: Population dynamics

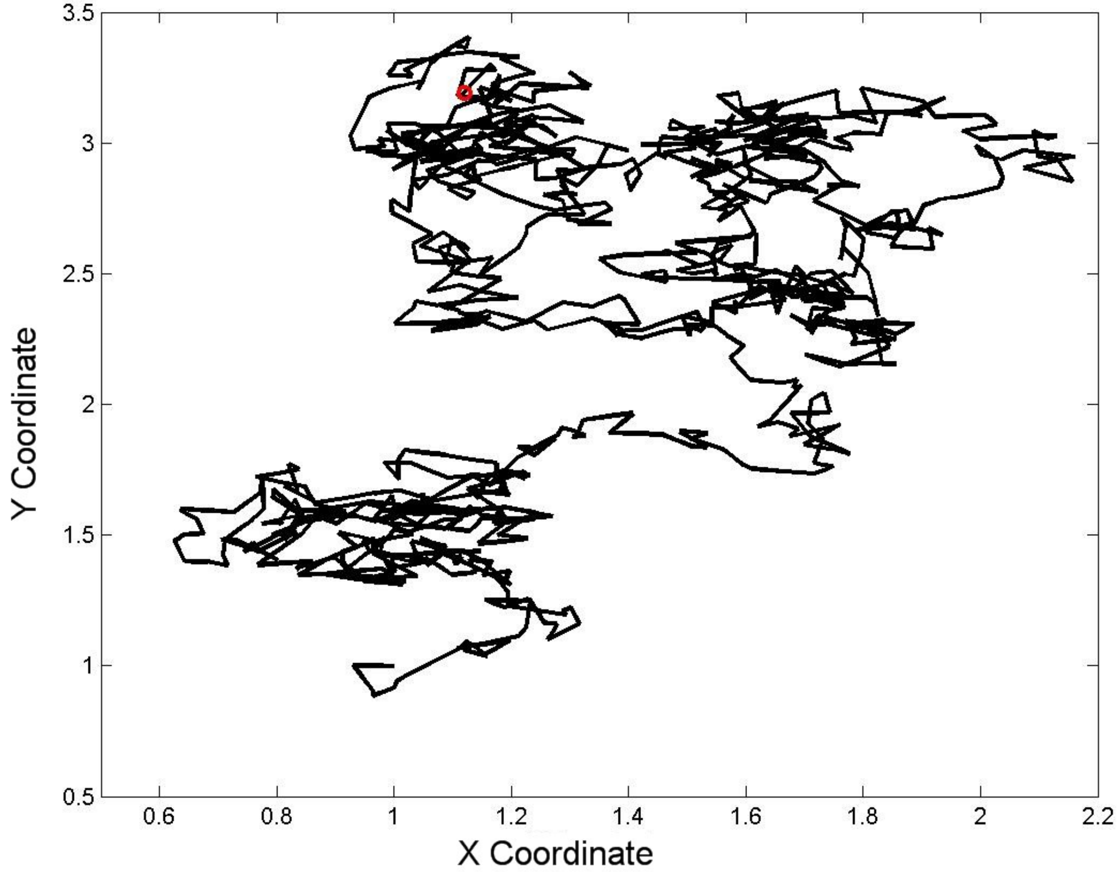
$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(\bar{K} - N(t)) + r\,N(t) \cdot \text{'noise'},\ t \in [0,T],\ \text{i.v. } N_0$$

How is the 'noise' modeled?

What is the mathematical definition of such an equation?

What can we say about the solutions of such a model?

Noise: Brownian motion of a particle in a fluid



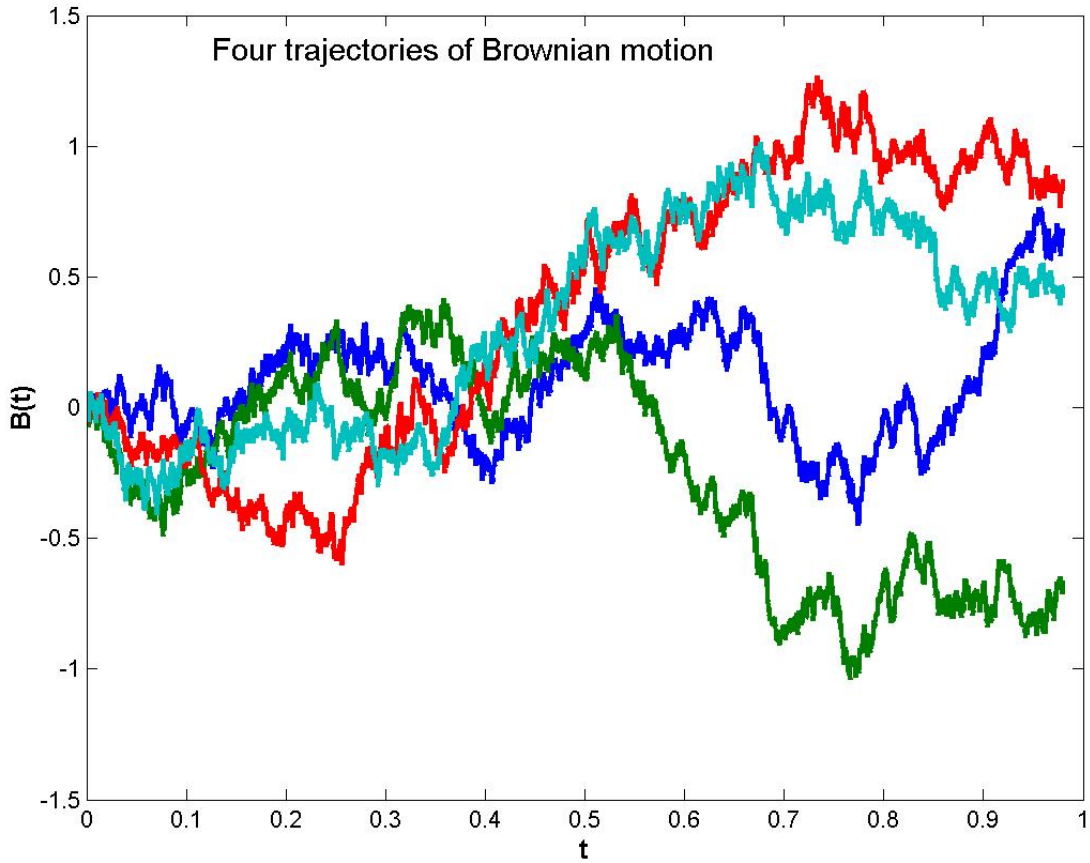
Brownian Motion

Brownian motion is a stochastic process $B(t, \omega)$, $\omega \in \Omega$ with

- $B(0) = 0$
- $B(t) - B(s)$ is a $\sqrt{t-s}\mathcal{N}(0,1)$ distr. random variable, $0 \le s < t \le T$,
- $B(t) - B(s)$, $B(v) - B(u)$ independent for $0 \le s \le t \le u \le v \le T$
- $B(t)$ has continuous paths
- $B(t)$ is nowhere differentiable
- total variation $= \infty$ on any finite interval $[0, T]$,

that is,

$$\sup_{\text{any partition}} \sum_{i=1}^{n} |B(t_i, \omega) - B(t_{i-1}, \omega)| = \infty$$


Four trajectories of Brownian motion
B(t)
t
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Some history

Jan Ingenhousz 1785

Robert Brown 1827

Louis Bachelier 1900

Albert Einstein 1905

Marian Smoluchowski 1906

Paul Langevin 1908

Norbert Wiener 1923

Kyosi Itô 1944, 1951

R.L. Stratonovich 1966

Motivation: Population dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} N(t) = r\, N(t)(\bar{K} - N(t)) + r\, N(t) \cdot \text{'noise'},\ t \in [0, T],\ \text{i.v.}\, N_0$$

'Noise' in this equation, coming from $K = \bar{K} +$ 'noise', is in general assumed to be 'white noise', the generalised derivative of Brownian motion.

However, Brownian motion is nowhere differentiable $\Rightarrow$ equation does not have a proper mathematical meaning!

$\Rightarrow$ use integrated version, need concept of stochastic integral $\int\limits_0^t f(s)\mathrm{d}B(s)$

Stochastic integral

(based on Mikosch, 1998)

grid $\{t_0 = 0, t_1, \ldots, t_N = 1\}$ on $[0, 1]$, $h =$max$(t_i - t_{i-1})$, $\tau_i \in [t_i, t_{i+1}]$

Riemann integral $\int\limits_0^t f(s)\mathrm{d}s = \lim\limits_{h \to 0} \sum\limits_{i=0}^{N-1} f(\tau_i)[t_{i+1} - t_i]$,

Riemann-Stieltjes integral $\int\limits_0^t f(s)\mathrm{d}g(s) = \lim\limits_{h \to 0} \sum\limits_{i=0}^{N-1} f(\tau_i)[g(t_{i+1}) - g(t_i)]$,

Take $g = B$?

Stochastic integral

Def: The real-valued function f on $[0,1]$ has bounded p-variation for some $p > 0$ if

$$\sup_{\text{any grid}} \sum_{i=0}^{N-1} |f(t_{i+1}) - f(t_i)|^p < \infty\,.$$

Result: The Riemann-Stieltjes integral $\int_0^t f(s)\mathrm{d}g(s)$ exists if

• the functions f and g do not have discontinuities at the same points $t \in [0,1]$

• the functions f, g have bounded p, q-variation resp. for some $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$.

Known: B has bounded q-variation on finite interval for $q > 2$

Stochastic integral

Result: f (deterministic or stochastic) differentiable with bounded derivative ($\Rightarrow$ bounded 1-variation) then the path-wise Riemann-Stieltjes integral $\int\limits_0^t f(s)\mathrm{d}B(s)$ exists.

Examples: $\int\limits_0^t e^s \mathrm{d}B(s)$, $\int\limits_0^t \sin(s)\mathrm{d}B(s)$, $\int\limits_0^t s^k \mathrm{d}B(s)$, $k \geq 0$.

Take $f = B$?

Stochastic integral

with partial integration: $\int\limits_a^b f(s)\mathrm{d}g(s) + \int\limits_a^b g(s)\mathrm{d}f(s) = f(b)g(b) - f(a)g(a)$

if $\int\limits_a^b B(s)\mathrm{d}B(s)$ were a Riemann-Stieltjes integral, we should have

$$\int\limits_a^b B(s)\mathrm{d}B(s) = \frac{1}{2}\int\limits_a^b B(s)\mathrm{d}B(s) + \frac{1}{2}\int\limits_a^b B(s)\mathrm{d}B(s) = \frac{1}{2}(B^2(b) - B^2(a))$$

Set $\tau_i = \lambda t_i + (1-\lambda)t_{i+1}$ and $B(\tau_i) = \lambda B(t_i) + (1-\lambda)B(t_{i+1})$, $\lambda \in [0,1]$

Result: on $[a,b]$ it holds

(conv. in L^2) $\quad \lim\limits_{h\to 0} \sum\limits_{i=0}^{N-1} (\lambda B(t_i) + (1-\lambda)B(t_{i+1}))[B(t_{i+1}) - B(t_i)]$

$= \frac{1}{2}(B^2(b) - B^2(a)) + (\lambda - \frac{1}{2})(b-a)$.

Itô Integral for $f(t)$ a stochastic process: (h grid-width)

$$\int_0^t f(s)\mathrm{d}B(s) = \quad \text{(conv. in } L^2\text{)} \quad \lim_{h\to 0}\sum_{i=0}^{N-1} f(t_i)[B(t_{i+1}) - B(t_i)]$$

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$\Uparrow$

evaluation at left endpoint !

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$\Uparrow$

evaluation at left endpoint !

Properties:

- usual linearity and additivity
- Martingale
- $\mathbb{E}(\int_0^t f(s)\mathrm{d}B(s)) = 0$
- $\mathbb{E}(|\int_0^t f(s)\mathrm{d}B(s)|)^2 = \mathbb{E}\int_0^t |f(s)|^2\mathrm{d}s$ "Itô isometry"

Remark: The latter properties are only valid for integrands with finite second moment, that is $\int_0^t \mathbb{E}(f^2(s))\mathrm{d}s < \infty$

Example: $f(t) = e^{B(t)}$, since $\mathbb{E}(\int_0^1 e^{2B(s)}\mathrm{d}B(s)) = \int_0^1 \mathbb{E}(e^{2B(s)})\mathrm{d}B(s) = \int_0^1 e^{2s}\mathrm{d}s = \frac{1}{2}(e^2 - 1) < \infty$, it holds that $\mathbb{E}(\int_0^1 e^{B(s)}\mathrm{d}B(s)) = 0$ and $\mathbb{E}(\int_0^1 e^{B(s)}\mathrm{d}B(s))^2 = \frac{1}{2}(e^2 - 1)$

Example: $f(t) = e^{B^2(t)}$, now $\int_0^1 \mathbb{E}(e^{2B^2(s)})\mathrm{d}s = \infty$, since $\mathbb{E}\big(e^{2B^2(s)}\big) = \int e^{2x^2}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dt = \infty$ for $t \geq 1/4$. Thus the Itô isometry does not hold and one can show that the expectation of the integral does not exist either.

(see Klebaner, 2005, for further examples)

Stochastic differential equations

defined as $\boxed{X(t) = X(0) + \int_0^t f(s, X(s))\ \mathrm{d}s + \int_0^t g(s, X(s))\ \mathrm{d}B(s)}$

abbreviated as $\boxed{\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t))\ \mathrm{d}B(t)}$

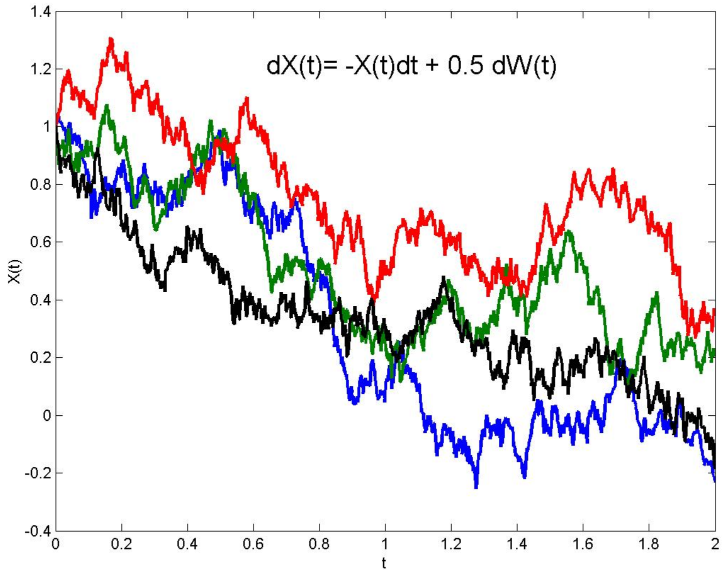
a Stochastic Verhulst model:

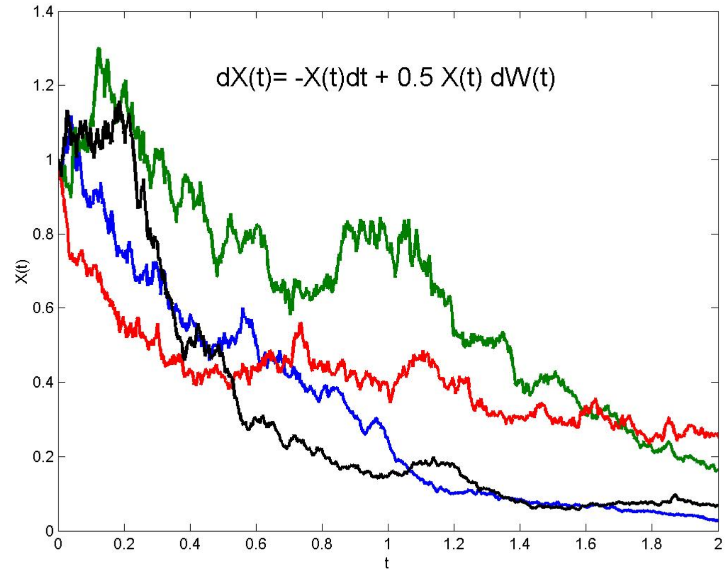
$$\mathrm{d}N(t) = \hat{r}\, N(t)(\bar{K} - N(t))\mathrm{d}t + \hat{r}\, N(t)\mathrm{d}B(t),\ t \in [0, T],\ \text{i.v.}\, N_0$$

Solutions of SODEs ($X(0) = 1$):

$\triangleright$ $\mathrm{d}X(t) = aX(t)\mathrm{d}t + b\mathrm{d}B(t)$, $\triangleright$ $X(t) = e^{at}(1 + b\int_0^t e^{-as}\mathrm{d}B(s))$

$\triangleright$ $\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}B(t)$, $\triangleright$ $X(t) = \exp((a - \frac{1}{2}b^2)t + bB(t))$





Itô formula = stochastic chain rule

Classical chain rule for f, g diffentiable

$$[f(g(t))]' = f'(g(t))g'(t)$$

or, in integral notation on $[0, t]$

$$f(g(t)) = f(g(0)) + \int_0^t f'(g(s))g'(s)\mathrm{d}s = f(g(0)) + \int_0^t f'(g(s))\mathrm{d}g(s)$$

$$\boxed{\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t))\ \mathrm{d}B(t)}$$

"Itô formula" (stochastic chain rule), for $\phi(x)$ function, suff. differentiable, scalar:

$$\boxed{\begin{aligned}\phi(X(t)) &= \phi(X(0)) + \int_0^t \phi'(X(s)) f(s, X(s)) \mathrm{d}s + \phi'(X(s)) g(s, X(s)) \mathrm{d}B(s) \\ &\quad + \frac{1}{2} \int_0^t \phi''(X(s))\ g^2(s, X(s))\ \mathrm{d}s\end{aligned}}$$

$$\phi(X(t)) = \phi(X(0)) + \int_0^t \phi'(X(s)) \mathrm{d}X(s) + \frac{1}{2} \int_0^t \phi''(X(s))\ g^2(s, X(s))\ \mathrm{d}s$$

Stratonovich integral

come back to

Set $\tau_i = \lambda t_i + (1-\lambda)t_{i+1}$ and $B(\tau_i) = \lambda B(t_i) + (1-\lambda)B(t_{i+1})$, $\lambda \in [0,1]$

Result: on $[a,b]$ it holds

(conv. in L^2) $\quad \lim_{h\to 0} \sum_{i=0}^{N-1} (\lambda B(t_i) + (1-\lambda)B(t_{i+1}))[B(t_{i+1}) - B(t_i)]$

$= \frac{1}{2}(B^2(b) - B^2(a)) + (\lambda - \frac{1}{2})(b-a)$.

$\lambda = \frac{1}{2} \Rightarrow$ integral follows deterministic calculus rules

Stratonovich integral

$$\int_0^t f(s) \circ \mathrm{d}B(s) = \quad \text{(conv. in } L^2) \quad \lim_{h \to 0} \sum_{i=0}^{N-1} f(\frac{t_i + t_{i+1}}{2})[B(t_{i+1}) - B(t_i)]$$

properties:

- usual linearity and additivity
- NOT Martingale
- $\mathbb{E}(\int_0^t f(s) \circ \mathrm{d}B(s)) \neq 0$
- Itô isometry does not hold
- conversion formula

Stratonovich SDEs

Stochastic differential equations

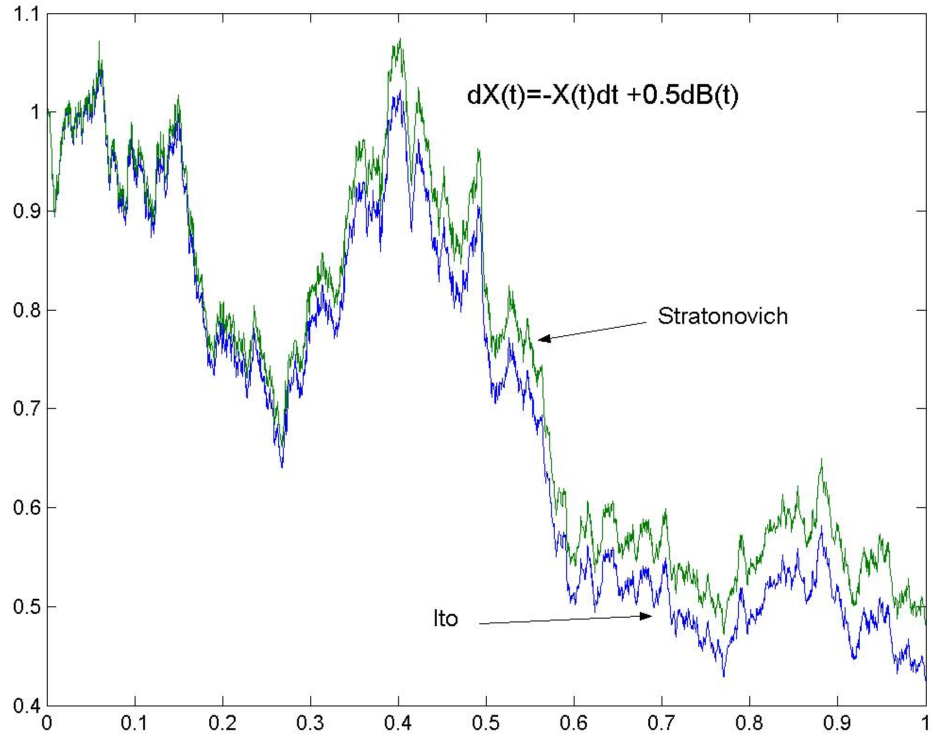
defined as $\boxed{X(t) = X(0) + \int_0^t f(s, X(s))\ \mathrm{d}s + \int_0^t g(s, X(s)) \circ \mathrm{d}B(s)}$

abbreviated as $\boxed{\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t)) \circ \mathrm{d}B(t)}$

Example:

$\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t)\mathrm{d}B(t)$ has solution $X(t) = \exp((a - \frac{1}{2}b^2)t + bB(t))$

$\mathrm{d}X(t) = aX(t)\mathrm{d}t + bX(t) \circ \mathrm{d}B(t)$ has solution $X(t) = \exp(at + bB(t))$


dX(t)=-X(t)dt +0.5dB(t)
Stratonovich
Ito
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Stratonovich SDEs

Using conversion formula, given Itô equation with solution X:

$$\mathrm{d}X(t) = f(t, X(t))\ \mathrm{d}t + g(t, X(t))\ \mathrm{d}B(t)$$

then $\int\limits_0^t \phi(s, X(s)) \circ \mathrm{d}B(s) = \int\limits_0^t \phi(s, X(s)) \mathrm{d}B(s) + \frac{1}{2} \int\limits_0^t g(s, X(s)) \phi_x(s, X(s)) \mathrm{d}s$

and with $\phi(s, X(s)) = g(s, X(s))$ the Stratonovich SODE corresponding to the above Itô equation with solution X is

$$\mathrm{d}X(t) = \tilde{f}(t, X(t))\ \mathrm{d}t + g(t, X(t)) \circ \mathrm{d}B(t)$$

where $\tilde{f}(t, X(t)) = f(t, X(t)) - \frac{1}{2} g(s, X(s)) g_x(s, X(s))$

Stratonovich SDEs

Brownian motion is an idealisation of physical noise, approximate B by B_n with B_n being a smoother, physically realisable processes, e.g. solutions of the Ornstein-Uhlenbeck/Langevin equation $\mathrm{d}X(t) = aX(t)\mathrm{d}t + b\mathrm{d}B(t)$.

Wong-Zakai (1965): Under certain conditions it holds with X_n and X satisfying

$$\mathrm{d}X_n(t) = f(t, X_n(t))\mathrm{d}t + g(t, X_n(t))\mathrm{d}B_n(t)$$

and

$$\mathrm{d}X(t) = (f(t, X(t)) + \tfrac{1}{2}g(t, X(t))\tfrac{\partial}{\partial x}g(t, X(t)))\mathrm{d}t + g(t, X(t))\mathrm{d}B(t)$$

that for $n \to \infty$ the solutions X_n converge to X, that is the solution of a Stratonovich equation.

Stability

Consider again the Verhulst model of population dynamics species

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)(K - N(t)) \qquad , \; t \in [0,T], \; \text{i.v.}\, N_0$$

Interested in steady-state solutions, i.e. when $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = 0$, in particular in positive, 'stable' steady-state solutions corresponding to 'survival' of the species.

Verhulst: two steady states $N_1 \equiv 0$ and $N_2 \equiv K$.

What is 'stability' and how do I determine if a steady-state is 'stable'?

Stability

Def.: The zero solution or steady state of an ODE $y'(t) = f(y(t))$ is termed

- *stable*, if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that the solution $y(t) = y(t; 0, y_0)$ exists for all $t \geq 0$ and $\boxed{|y(t)| < \epsilon}$ whenever $t \geq 0$ and $|y_0| < \delta$;

- *asymptotically stable*, if it is stable and if there exists a $\delta \geq 0$ such that whenever $|y_0| < \delta$ we have $\boxed{|y(t)| \to 0 \quad \text{for} \quad t \to \infty.}$

For linear $y'(t) = \lambda y(t)$ with solution $y(t) = y_0\, e^{\lambda t}$ it is obvious that the zero solution (starting with $y_0 = 0$) is asymptotically stable iff $\lambda < 0$.

Stability

For nonlinear ODE $y'(t) = f(y(t))$ use 'linearisation approach':

Given a steady-state solution $\bar{y}$, expand f around it:

$f(y(t)) = f(\bar{y}) + f'(\bar{y})(y(t) - \bar{y}) +$ higher order terms

insert into ODE and rearrange for new $x(t) = y(t) - \bar{y}$ to get $\big(f(\bar{y}) = 0$ as $\bar{y}$ is a steady state)

$x'(t) = f'(\bar{y})x(t) +$ higher order terms

Now the zero solution of this and thus the steady state solution of the original is is asymptotically stable iff $f'(\bar{y}) < 0$.

Remark: if $f'(\bar{y}) = 0$ we can not neglect the higher order terms and can not conclude the stability behaviour this way.

Asymptotic MS-stability of zero solutions

Def.: The zero solution of an SODE is termed

- *mean-square stable*, if for each $\epsilon > 0$, there exists a $\delta \geq 0$ such that the solution $X(t) = X(t; 0, X_0)$ exists for all $t \geq 0$ and $\boxed{\mathbb{E}|X(t)|^2 < \epsilon}$ whenever $t \geq 0$ and $\mathbb{E}|X_0|^2 < \delta$;

- *asymptotically mean-square stable*, if it is mean-square stable and if there exists a $\delta \geq 0$ such that whenever $\mathbb{E}|X_0|^2 < \delta$ we have $\boxed{\mathbb{E}|X(t)|^2 \to 0 \quad \text{for} \quad t \to \infty.}$

Linear equation, for $t \geq 0$, with $X(0) = X_0, \quad \lambda, \mu, X_0 \in \mathbb{R}$,

$$\boxed{\mathrm{d}X(t) = \lambda\, X(t)\mathrm{d}t \;+\; \mu\, X(t)\mathrm{d}B(t), \qquad (1)}$$

with the geometric Brownian motion $X(t) = \exp((\lambda - \frac{1}{2}\mu^2)t + \mu B(t))$ as exact solution.

Thm.: (e.g. in Arnold '74, Hasminskii '80)
The zero solution of (1) is asymptotically mean-square stable if

$$\boxed{\lambda < -\tfrac{1}{2}\, |\mu|^2}$$

References

[1] L. Arnold. *Stochastic Differential Equations: Theory and Applications*. Wiley-Interscience, New York, 1974.

[2] T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel-Dekker, New York, 1988.

[3] R. Z. Hasminskiĭ, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, 1981.

[4] F. C. Klebaner, *Introduction to stochastic calculus with applications (2nd ed.)*, Imperial College Press, London, 2005.

[5] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing Limited, Chichester, 1997.

[6] T. Mikosch, *Elementary stochastic calculus with finance in view*. World Scientific, Singapore, 1998.

[7] E. Wong and M. Zakai, On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Stat.* 36, pp. 1560–1564, 1965.