Fractional Brownian motion and applications

Part I: fractional Brownian motion in Finance

INTRODUCTION

The fBm is an extension of the classical Brownian motion that allows its disjoint increments to be correlated.

Motivated by empirical studies, several authors have studied financial models driven by the fBm.

Fractional stochastic volatility models

Fractional Black-Scholes model

INTRODUCTION

Fractional stochastic volatility models (see Comte and Renauld (1998) or Comte, Coutin and Renault (2003) explain better the long-time behaviour of the implied volatility.

Nevertheless, the fBm (and then the volatility) are not Markovian, and this becomes a strong difficulty to study and to put these models into practice (the usual techniques assume the Markov property).

INTRODUCTION

The introduction of the fractional Black-Scholes model, where the Brownian motion in the classical Black-Scholes model is replaced by a fBm, have been motivated by empirical studies (see for example Mandelbrot (1997), Shiryaev (1999) or Willinger (1999)).

Unfortunately, they allow for arbitrage opportunities (see for example Cheridito (2003) and Sottinen (2001)). This cashm between theory and practice have been the motivation of several works that have tried to preserve the fBm approach at the same time they exclude the arbitrage opportunities:

INTRODUCTION

Elliot and Van der Hoek (2003) or Hu and Oksendal (2003) suggested models where the classical integrals were substituted by integrals in the Wick sense. These models have not arbitrage opportunities but, as it was proved in Bjork and Hult (2005), they have no natural economic interpretation.

Cheridito (2003) proves that the arbitrage opportunities disappear by introducing a minimal ammount of time between transactions. Guasoni (2005) proves that they also disappear under transaction costs. These papers open a very interesting field of research.

THE FRACTIONAL BROWNIAN MOTION

A centered Gaussian process B^H is called a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ if it has the covariance function

$$\boxed{R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)}$$

Usually it is assumed that $B_0^H = 0$.

THE FRACTIONAL BROWNIAN MOTION

Basic properties

If $H = 1/2$, $B^{1/2}$ is a standard Brownian motion

It is self-similar:

for $a > 0$, the law of $a^{-H} B_{at}^{H}$

is the same as the law of B_t^H

If $H \neq 1/2$, B^H is not a semimartingale

$$E\left[\left(B_t^H - B_s^H\right)^2\right] = (t-s)^{2H}$$

THE FRACTIONAL BROWNIAN MOTION

$$\text{If } H > 1/2$$

$$\Rightarrow \text{ disjoint increments positively correlated:}$$

$$E\left[\left(B_t^H - B_s^H\right)\left(B_s^H - B_r^H\right)\right] > 0$$

$$\text{If } H < 1/2$$

$$\Rightarrow \text{disjoint increments negatively correlated}$$

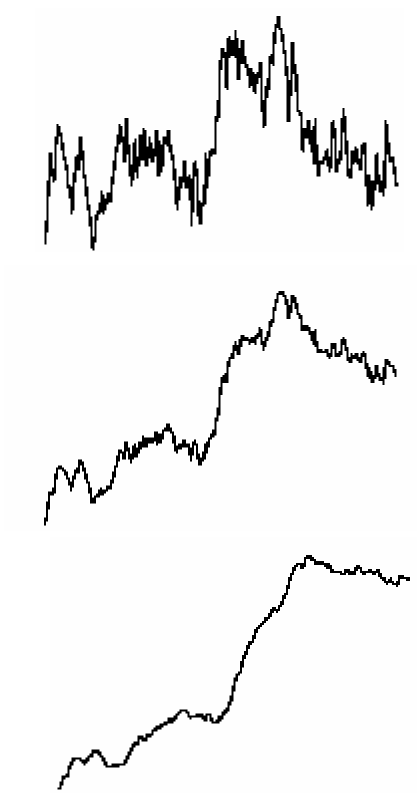
$$E\left[\left(B_t^H - B_s^H\right)\left(B_s^H - B_r^H\right)\right] < 0$$

λ - Hölder continous, for every $\lambda < H$

THE FRACTIONAL BROWNIAN MOTION

Simulation of a typical path of fBm:

(from Cheridito (2001))

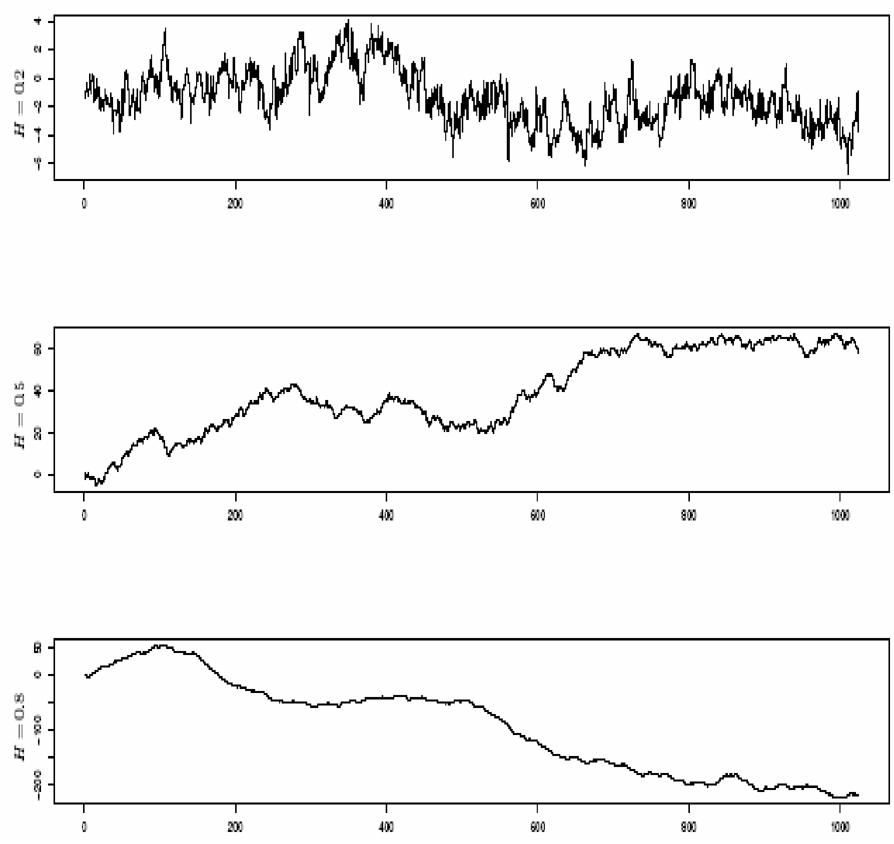


H=0.2

H=0.5

H=0.8

THE FRACTIONAL BROWNIAN MOTION



H=0.2

H=0.5

H=0.8

(from Dieker (2004))

THE FRACTIONAL BROWNIAN MOTION

Representations

Mandelbrot and Van Ness (1968):

$$B_t^H = \frac{1}{C_1(H)} \int_R \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}} \right] dW_s,$$

$$\text{where } C_1(H) = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right) ds + \frac{1}{2H} \right)^{\frac{1}{2}}$$

THE FRACTIONAL BROWNIAN MOTION

Other representations (see for example Nualart (2003))

$$B_t^H = \int_0^t K_H(t,s)\,dW_s,$$

<u>Case $H > 1/2$</u>

$$\Rightarrow K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where

$$c_H = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)} \right]^{\frac{1}{2}}$$

THE FRACTIONAL BROWNIAN MOTION

Case $H < 1/2$

$$\Rightarrow K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right]$$

where

$$c_H = \left[\frac{2H}{(1-2H)\beta\left(1-2H, H+\frac{1}{2}\right)} \right]^{\frac{1}{2}}$$

THE FRACTIONAL BROWNIAN MOTION

Some works (as Alòs, Mazet and Nualart (2001) or Comte and Renault (1998)) deal with the following truncated version of the fractional Brownian motion:

$$W_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

This process is not a fBm, but it has a simpler representation while it preserves most of the basic properties of the fBm.

STOCHASTIC CALCULUS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

$$H \neq 1/2 \Rightarrow B^H \text{ is not a semimartingale}$$
$$\Rightarrow \text{We can not apply classical Itô's calculus}$$

<u>Possible approaches</u>

Pathwise techniques

(Zähle (1998))

Malliavin calculus techniques

(Carmona, Coutin and Montseny (2003), Alòs, Mazet and Nualart (2000))

Integration of deterministic functions

We denote by H the Hilbert space with scalar product defined by

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{H} = R_H(t,s)$$

The mapping $1_{[0,t]} \rightarrow B_t^H$ can be extended to an isometry between H and the Gaussian space $H_1\left(B^H\right)$ associated with B^H. We denote this isometry $\varphi \rightarrow B^H(\varphi)$.

STOCHASTIC CALCULUS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

$$H > 1/2:$$

$$R_H(t,s) = H(2H-1)\int_0^t \int_0^2 |r-u|^{2H-2}\,du\,dr$$

$$\Downarrow$$

$$\langle \varphi, \psi \rangle = H(2H-1)\int_0^T \int_0^T |r-u|^{2H-2}\varphi_r \psi_u\,du\,dr$$

Then we deduce the representation

$$B^H(\varphi) = \int_0^T \left(\int_s^T \frac{\partial K_H}{\partial r}(r,s)\varphi(r)\,dr \right) dW_s$$

STOCHASTIC CALCULUS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

In the case H<1/2, similar arguments give us that

$$B^H(\varphi) = \int_0^T \left[K_H(T,s)\varphi(s) + \int_s^T \frac{\partial K_H}{\partial r}(r,s)\left(\varphi(r) - \varphi(s)\right) dr \right] dW_s$$

STOCHASTIC CALCULUS WITH RESPECT TO THE FRACTIONAL BROWNIAN MOTION

Pathwise integrals in the case H>1/2

Suppose that f, g are Hölder continous functions of orders α and β, with $\alpha + \beta > 1$. Then the Riemann-Stieltjes integral $\int fdg$ exists

$\Downarrow$

If $H > 1/2$ and F is regular enough, $\int F(B^H{}_s)dB_s^H$ exists (in the Riemann-Stieltjes sense). Moreover

$$F\left(t, B_t^H\right) = F(0,0) + \int_0^t \frac{\partial F}{\partial x}\left(s, B_s^H\right)ds$$

$$+ \int_0^t \frac{\partial F}{\partial x}\left(s, B_s^H\right)dB_s^H$$

APPLICATIONS IN FINANCE

Models driven by the fBm: the arbitrage problem

Consider the fractional Black-Scholes model for a bond (X_t) and a stock (Y_t) (H>1/2):

$$X_t = \exp(rt)$$

$$Y_t = Y_0 \exp\left[(r+\nu)t + \sigma B_t^H\right]$$

The introduction of this model has been motivated by empirical studies (see for example Willinger et al. (1999))

APPLICATIONS IN FINANCE

This model gives arbitrage opportunities. For example, we can take

$$\vartheta_t^0 := cY_0\left[1-\exp\left(2\nu t+2\sigma B_t^H\right)\right]$$

$$\vartheta_t^1 := 2c_0\left[\exp\left(2\nu t+2\sigma B_t^H\right)-1\right]$$

Then, Itô's formula gives us that

$$\vartheta_t^0 X_t + \vartheta_t^1 Y_t$$

$$= \vartheta_0^0 X_0 + \vartheta_0^1 Y_0 + \int_0^t \vartheta_u^0 dX_u + \int_0^t \vartheta_u^1 dY_u$$

$$= cY_0 \exp(rt)\left\{\exp\left(2\nu t+\sigma B_t^H\right)-1\right\}^2$$

$$\Downarrow$$

$$\left(\vartheta^0, \vartheta^1\right) \text{is an arbitrage self - financing strategy}$$

APPLICATIONS IN FINANCE

Cheridito (2003) proved that, even the market allows for arbitrage strategies, these strategies cannot be constructed in practice. In fact, he proved that if there is a mimimum ammount of time between transactions, the arbitrage opportunities disappear. The main idea is the following:

For the sake of simplicity, we assume $\nu = 0$

(and then $\underbrace{\tilde{Y}_t}_{\text{actualized value}} = Y_0 \exp\left(B_t^H\right)$)

APPLICATIONS IN FINANCE

Consider the strategy defined by

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{i=1}^{n-1} g_i 1_{(\tau_i, \tau_{i+1}]}$$

$$\text{where } \tau_{i+1} - \tau_i > h$$

$$\left(\vartheta^0, \vartheta^1\right) \text{is self-financing}$$

$$\Downarrow$$

$$\tilde{V}_T = \tilde{V}_0 + \left(\vartheta^1 \cdot \tilde{Y}\right) = \sum g_i \left(\exp\left(B^H_{\tau_{i+1}}\right) - \exp\left(B^H_{\tau_i}\right)\right)$$

$$\downarrow$$

actualized value

APPLICATIONS IN FINANCE

Assume that this strategy allows for arbitrage and let k be the first moment l such that

$$\sum_{i=1}^{l} g_i \left(\exp\left(B^H_{\tau_{i+1}}\right) - \exp\left(B^H_{\tau_i}\right) \right) > 0 \; a.s.$$

Contradiction!!!!

Notice that

$$\sum_{i=1}^{k} g_i \left(\exp\left(B^H_{\tau_{i+1}}\right) - \exp\left(B^H_{\tau_i}\right) \right)$$

$$= \sum_{i=1}^{k-1} g_i \left(\exp\left(B^H_{\tau_{i+1}}\right) - \exp\left(B^H_{\tau_i}\right) \right) \quad \boxed{\leq 0}$$

$$+ g_k \left(\exp\left(B^H_{\tau_{k+1}}\right) - \exp\left(B^H_{\tau_k}\right) \right)$$

It can be <0!!

It can be <0!!

APPLICATIONS IN FINANCE

Guasoni (2006) proved that the arbitrage opportunities also disappear under transaction costs. To achieve an arbitrage, at some point t_0 we have to start trading. This decision generates a transaction cost which must be recovered at a latter time, and this is possible only if the asset price moves enough in the future. Hence, if at all times there is a remote possibility of arbitrary small price changes, then downside risk cannot be eliminated,and arbitrage is impossible.

The above results by Cheridito (2003) and Guasoni (2006) open a new scenario, where the fBm can be an appropiate for stock price modelling if we assume that the non-existence of arbitrage strategies is not due to the market, but to the existence of restrictions on the trading strategies.

APPLICATIONS IN FINANCE

Long-memory stochastic volatility models

Stochastic volatility models:

$$dS_t = rS_t dt + \sigma_t S_t dW_t$$

Stochastic process

(see for example Heston (1993), Hull and White (1987), Stein and Stein (1991) or Scott (1987))

If the volatility is not correlated with W, these models deal to a symmetric implied volatility smile (see Renault and Touzi (1996))

A asymmetric implied volatility skew can be explained by the existence of a negative correlation between W and the volatility process.

APPLICATIONS IN FINANCE

Nevertheless, the dependence of the implied volatility on time to maturity (term structure) is not well explained by classical stochastic volatility models.

In practice, de decreasing of the smile amplitude when time to maturity increases turns out to be much slower than it goes according to stochastic volatility models.

With this aim, Comte and Renault (1998) and Comte, Coutin and Renault (2003) have proposed stochastic volatility models based on the fBm. These models allows us to explain the observed long-time behaviour of the implied volatility.

APPLICATIONS IN FINANCE

In Comte and Renault (1998) the volatility process is given by

$$\sigma_t = f\left(Y_t\right), \text{where}$$

$$Y_t = m + \left(Y_0 - m\right)e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dB_s^H$$

uncorrelated with W
H>1/2

In this context, the classical Hull and White formula gives us that call option prices can be written as

APPLICATIONS IN FINANCE

$$V_t = E_Q\left[C_{BS}\left(t, S_t; \frac{1}{T-t}\int_t^T \sigma_s^2 ds \right) \middle| F_t \right]$$

Classical Black-Scholes formula

Risk-neutral probability

Then, the authors state that the dynamics of the implied volatilty are directly related to the dynamics of

$$u_t := \frac{1}{T-t}\int_t^T \sigma_s^2 ds$$

Notice that $Cov(u_t, u_{t+h}) = O(h^{2H-2}), h \to \infty$

(this does not vanish at the exponential rate, but at the hyperbolic rate, which explains the long-time behaviour of stochastic volatilities)

APPLICATIONS IN FINANCE

A recent paper of Comte, Coutin and Renault (2003) deal with a stochastic volatility process of the form :

$$\sigma_t^2 = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \tilde{\sigma}_s^2 \, ds,$$

where $\tilde{\sigma}_s$ is a square root process

As $\tilde{\sigma}_s$ is Markovian, this long - memory model becomes simpler from the computational point of view.

APPLICATIONS IN FINANCE

In resume, fractional stochastic volatility models allow us to explain the long-time behaviour of the implied volatility, but they are more complex and new technical difficulties arise.

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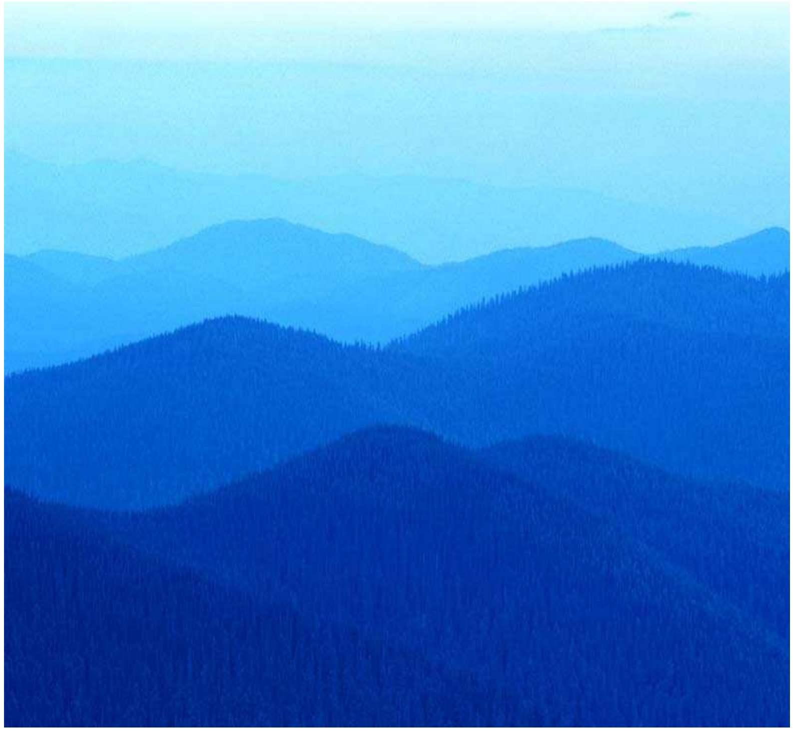
Fractional Brownian motion and applications

Part II: Applications to surface growth modelling

INTERFACES IN NATURE

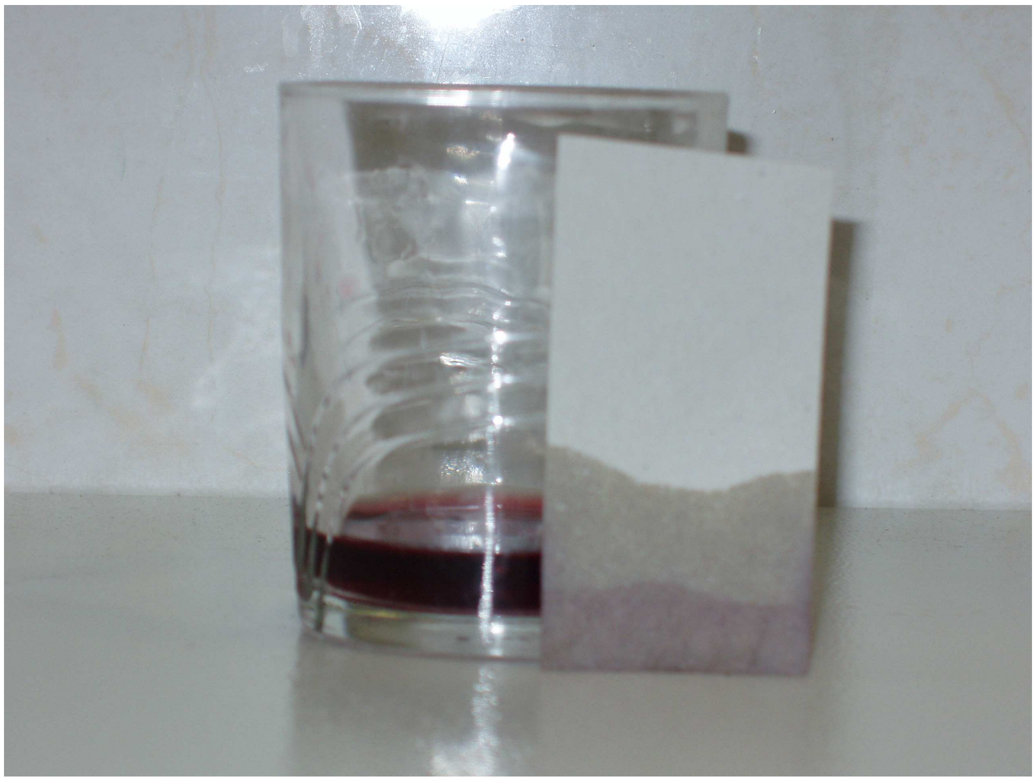
Most of our life takes place on the surface of something:

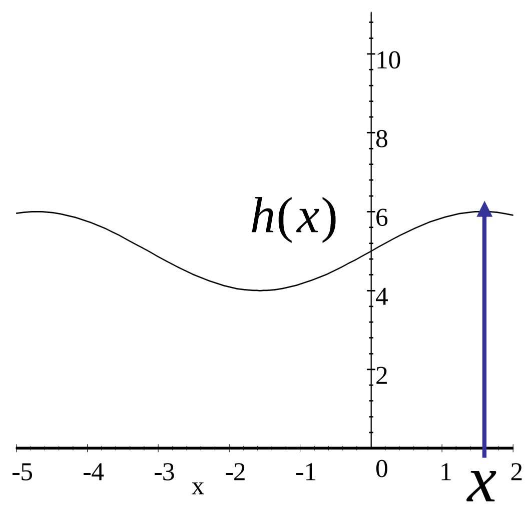


Interesting questions:

formation, growth and dynamics

SOME EXAMPLES (I)





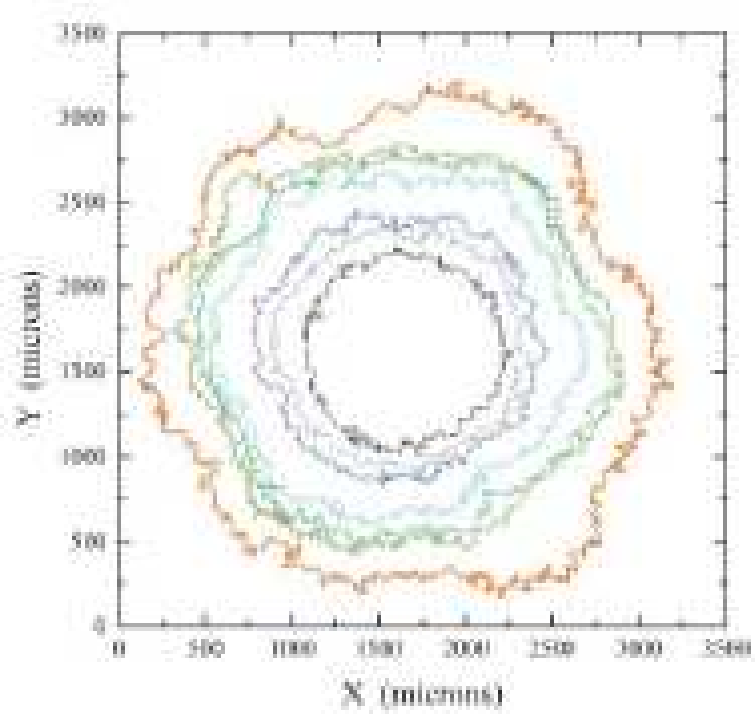
SOME EXAMPLES (II)

combustion

particle deposition

SOME EXAMPLES (III)

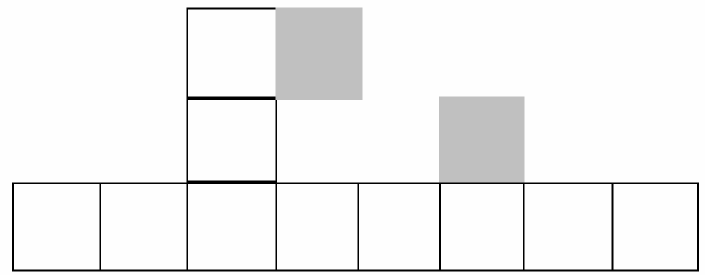
Radial symmetry



tumor growth

(Bru et al., Biophysical Journal 2003)

BASIC SCALING NOTIONS (I)



Ballistic deposition

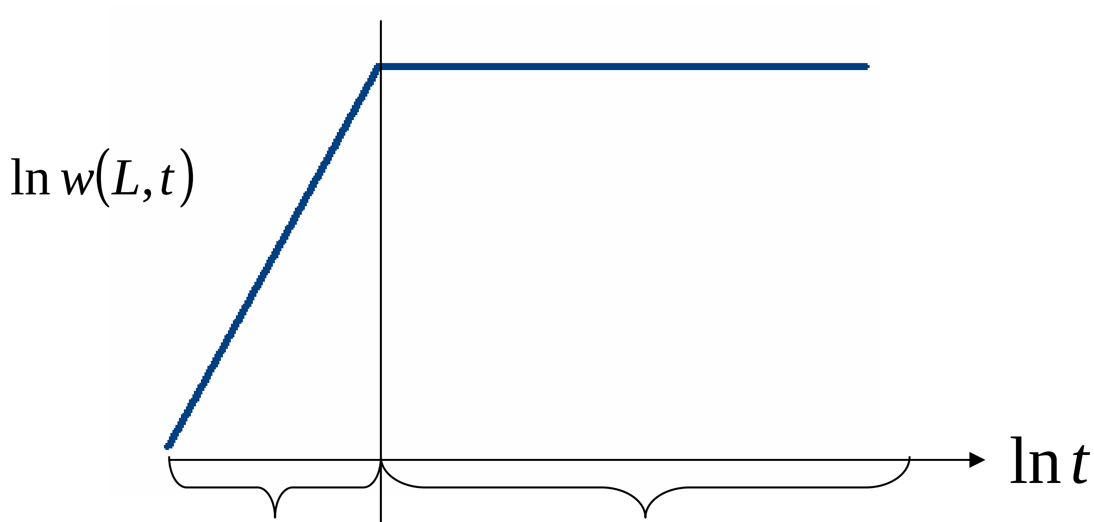
Roughness:

Mean height $$\bar{h}(t) = \frac{1}{L}\sum_{i=1}^{L} h(i,t)$$

Interface width (roughness) $$w(L,t) = \sqrt{\frac{1}{L}\sum_{i=1}^{L}\left[h(i,t) - \bar{h}(t)\right]^2}$$

BASIC SCALING NOTIONS (II)

A typical plot of the time evolution of the surface width



$w(L,t) = t^{\beta}$ $\quad$ $w(L,t) = L^{\alpha}$ (saturation due by correlation

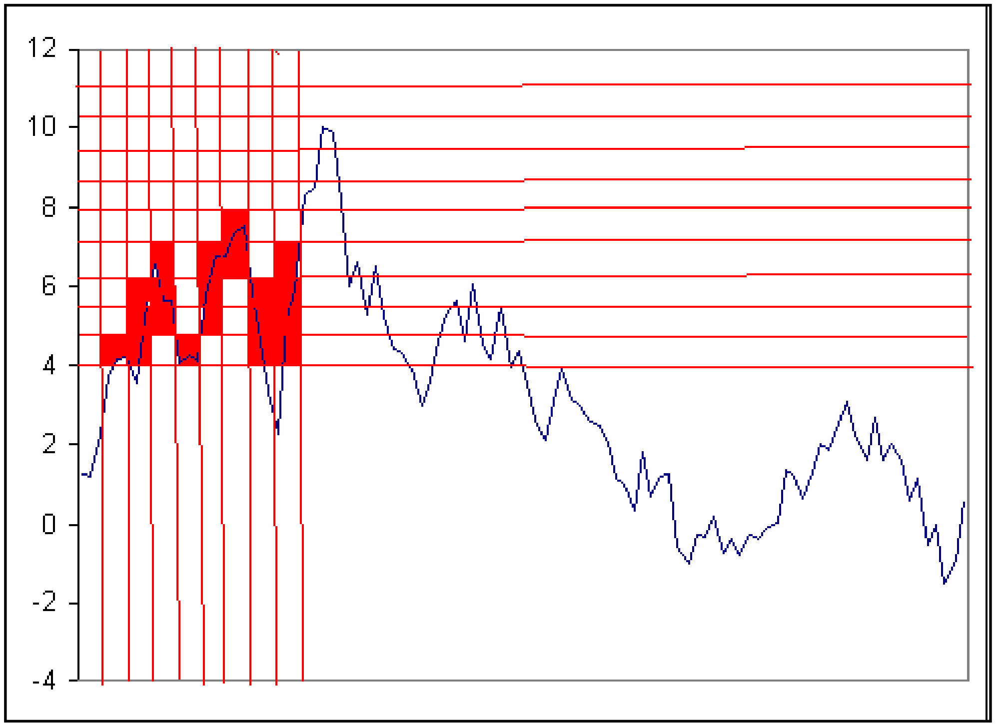
$$t_x \approx L^z$$

$$\ln t_x \approx z \ln L$$

$$\boxed{z = \frac{\alpha}{\beta}; w(L,t) \approx L^{\alpha} f\left(\frac{t}{t_x}\right)}$$

NOTIONS ON FRACTAL GEOMETRY (I)

Fractal dimension



$$d_f = \lim_{l \to 0} \frac{\ln N(l)}{\ln(1/l)}$$

NOCIONS DE GEOMETRIA FRACTAL (II)

Self-affinity (exact or statistical)

$$h(x) \approx b^{-\alpha} h(bx)$$

Fractal dimension and self-affinity (exact or statistical)

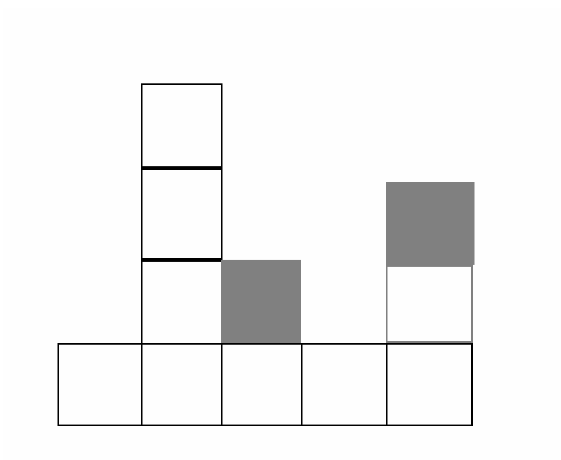
$$\Delta(l) \equiv \left| h(x_1) - h(x_2) \right| \approx l^{\alpha}$$

$$\left| x_1 - x_2 \right| \equiv l$$

and then $d_f = 2 - \alpha$

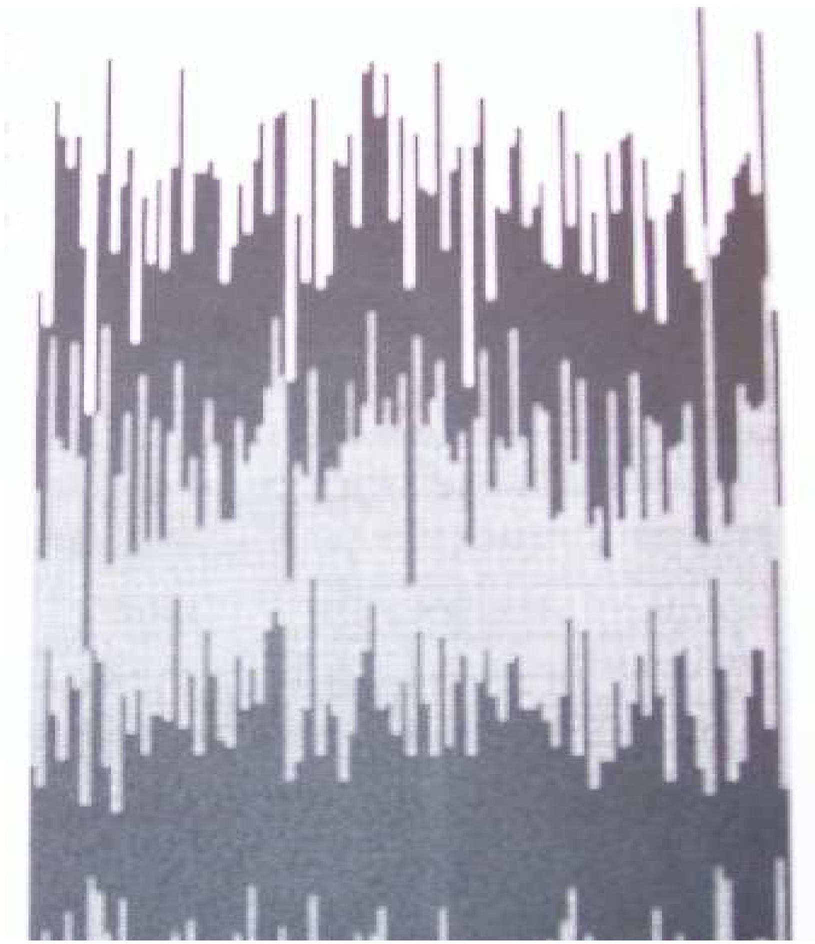
NOTIONS ON MODELLING (I)

Random deposition



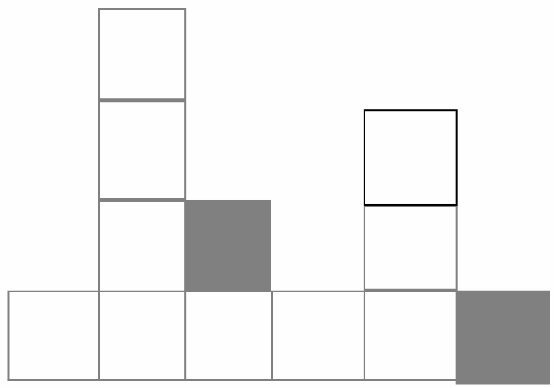
$$\frac{\partial h}{\partial t} = F + dW_{t,x}$$

$$\beta = 1/2$$



NOTIONS ON MODELLING (II)

Random deposition with surface relaxation



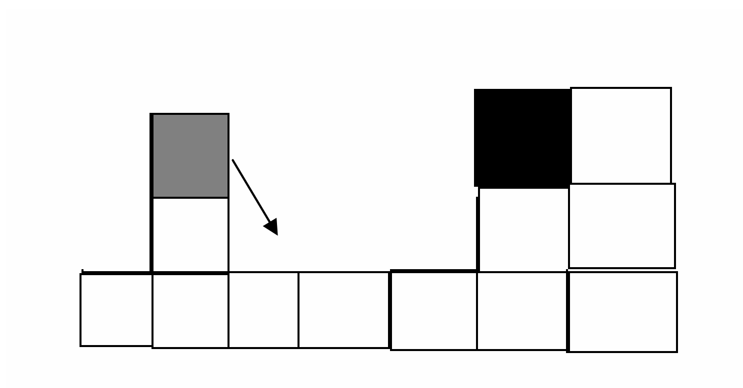
$$\frac{\partial h}{\partial t} = F + \frac{\partial^2 h}{\partial x^2} + dW_{t,x}$$

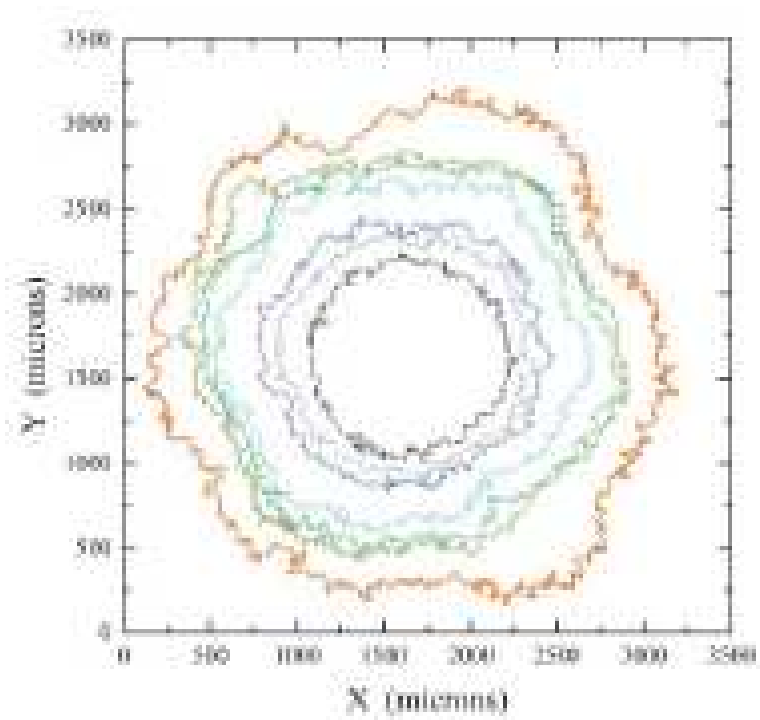
$$\alpha = 1/2, \beta = 1/4, z = 2$$

(Edward-Wilkinson)

NOTIONS ON MODELLING (III)

Molecular beam epitaxy (MBE)





$$\frac{\partial h}{\partial t} = F + \frac{\partial^4 h}{\partial x^4} + dW_{t,x}$$

$$\alpha = 3/2, \beta = 3/8, z = 4$$

(MBE)

CORRELATED NOISE (FBM)

$$\langle \eta(t,x), \eta(t,x) \rangle \approx |x - x'|^{\varphi - 1} \delta(t - t')$$

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